

Improving Christofides' Algorithm for the s - t Path TSP

Hyung-Chan An

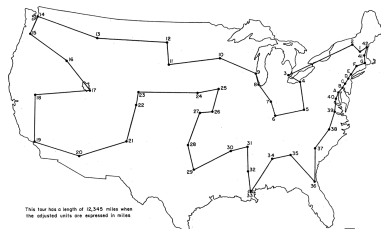
Cornell University

May 21, 2012

Joint work with Bobby Kleinberg and David Shmoys

Metric TSP

- Metric (circuit) TSP
 - Given a weighted graph $G = (V, E)$ ($c : E \rightarrow \mathbb{R}_+$), find a minimum Hamiltonian circuit
 - Triangle inequality holds
 - Christofides (1976) gave a $3/2$ -approximation algorithm



This tour has a length of 12,345 miles when the adjusted units are expressed in miles.

FIG. 16. The optimal tour of 49 cities.

Figure from [Dantzig, Fulkerson, Johnson 1954]

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 - Given a weighted graph $G = (V, E)$ ($c : E \rightarrow \mathbb{R}_+$), find a minimum Hamiltonian circuit
 - Triangle inequality holds
 - Christofides (1976) gave a $3/2$ -approximation algorithm
 - No better performance guarantee known

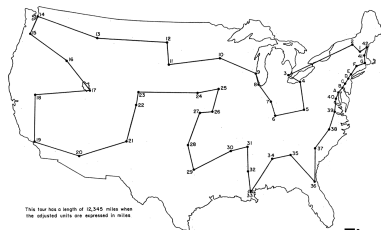


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Metric s - t Path TSP

- Metric s - t path TSP
 - Given a weighted graph $G = (V, E)$ ($c : E \rightarrow \mathbb{R}_+$) with endpoints $s, t \in V$, find a minimum s - t Hamiltonian path
 - Triangle inequality holds
 - Hoogeveen (1991) showed that Christofides' algorithm is a $5/3$ -approximation algorithm and this bound is tight

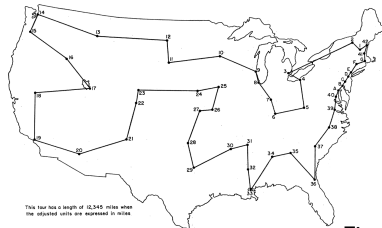


FIG. 16. The optimal tour of 49 cities.

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Our Main Result

Theorem

Christofides' algorithm can be improved to yield a deterministic ϕ -approximation algorithm for the s-t path TSP for an arbitrary metric, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio ($\phi < 1.6181$)

Recent Exciting Improvements

- Recent improvements for **unit-weight graphical metric TSP**
 - Shortest path metric in an underlying unweighted graph
 - Better approximation than Christofides' ([Oveis Gharan, Saberi, Singh 2011], [Mömke, Svensson 2011], [Mucha 2011], [Seboř, Vygen 2012])

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 - Better approximation than Christofides' ([Oveis Gharan, Saberi, Singh 2011], [Mömke, Svensson 2011], [Mucha 2011], [Seboř, Vygen 2012])
 - Techniques can be successfully applied to both variants
- Our algorithm for the s - t path TSP improves Christofides' for an **arbitrary** metric
 - Can our techniques be extended to the circuit variant?

Can Randomization Beat Christofides?

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- Augment \mathcal{I}_{\min} into a low-cost Eulerian circuit/path
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 - $O(\log n / \log \log n)$ -approx for ATSP
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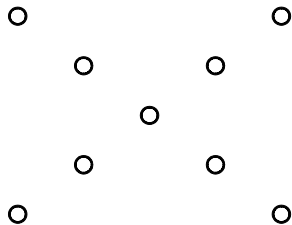
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- ϕ -approx for s - t path TSP
 - Arbitrary metric
 - Simpler random choice

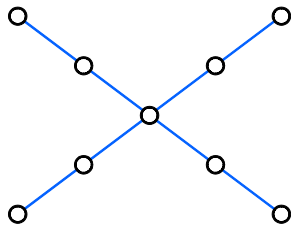
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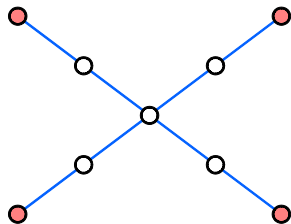
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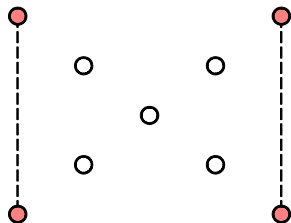
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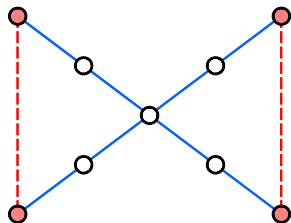
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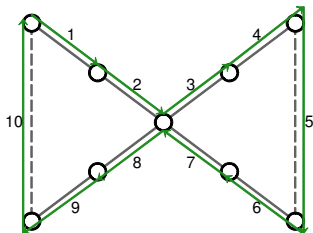
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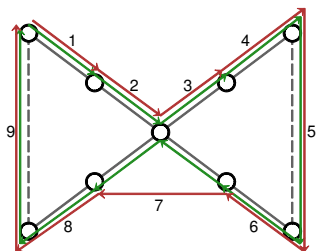
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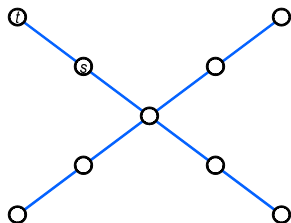
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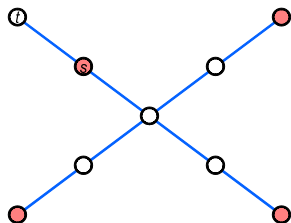
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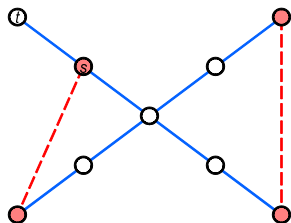
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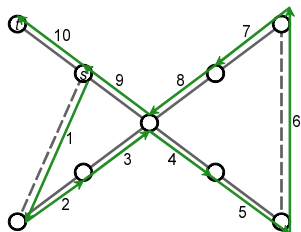
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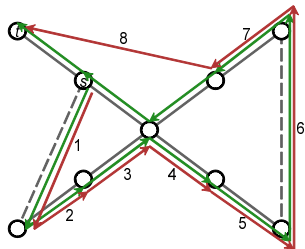
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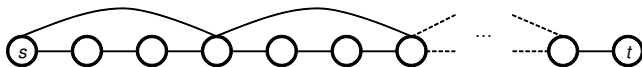
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Path-variant Christofides' algorithm

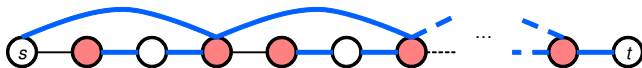
- Path-variant Christofides' algorithm
 - 5/3-approximation algorithm [Hoogeveen 1991]
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distance between two vertices defined as shortest distance on this underlying unit-weight graph

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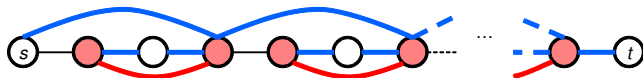
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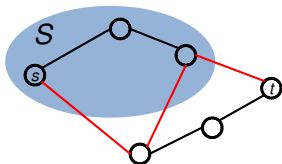
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Held-Karp Relaxation

- Held-Karp relaxation
 - $\delta(S)$ for $S \subsetneq V$ denotes the set of edges in cut (S, \bar{S})



- Incidence vector χ_F of $F \subset E$ is $(\chi_F)_e := \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$

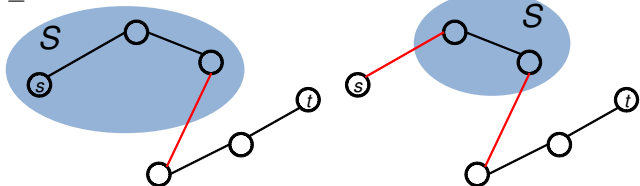
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For $G = (V, E)$ and $s, t \in V$,

$$\left\{ \begin{array}{l} \sum_{e \in \delta(\{s\})} x_e = \sum_{e \in \delta(\{t\})} x_e = 1 \\ \sum_{e \in \delta(\{v\})} x_e = 2, \quad \forall v \in V \setminus \{s, t\} \\ \sum_{e \in \delta(S)} x_e \geq 1, \quad \forall S \subsetneq V, |\{s, t\} \cap S| = 1 \\ \sum_{e \in \delta(S)} x_e \geq 2, \quad \forall S \subsetneq V, |\{s, t\} \cap S| \neq 1, S \neq \emptyset \\ 0 \leq x_e \leq 1 \end{array} \right. \quad \forall e \in E$$

$x \in \mathbb{R}^E$



Held-Karp Relaxation

- Polynomial-time solvable
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- Feasible region of this LP is contained in the ST polytope
- Held-Karp solution can be written as a convex combination of (incidence vectors of) spanning trees
- Can find such a decomposition in polynomial time [Grötschel, Lovász, Schrijver 1981]

Our Algorithm

- Best-of-Many Christofides' Algorithm
 - *Compute an optimal solution x^* to the Held-Karp relaxation*
 - *Rewrite x^* as a convex comb. of spanning trees $\mathcal{T}_1, \dots, \mathcal{T}_k$*
 - For each \mathcal{T}_i :
 - Let T_i be the set of vertices with “wrong” parity of degree:
i.e., T_i is the set of even-degree endpoints and other odd-degree vertices in \mathcal{T}_i
 - Find a minimum perfect matching M_i on T_i
 - Find an s - t Eulerian path of $\mathcal{T}_i \cup M_i$
 - Shortcut it into an s - t Hamiltonian path H_i
 - Output the best Hamiltonian path

Randomized Algorithm

- Sampling Christofides' Algorithm
 - Sample \mathcal{T} by choosing \mathcal{T}_i with probability λ_i
($x^* = \sum_{i=1}^k \lambda_i \chi_{\mathcal{T}_i}$)

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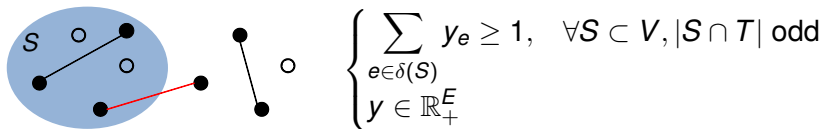
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Best-of-Many Christofides' Algorithm is ρ -approx. algorithm
- $\Pr[e \in \mathcal{H}] = x_e^*$
 - $E[c(\mathcal{H})] = \sum_{e \in E} c_e x_e^* = c(x^*)$
 - The rest of the analysis focuses on bounding $c(M)$

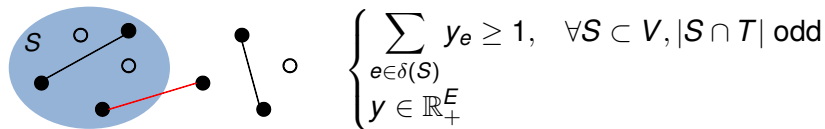
Polyhedral Characterization of Matchings

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- Call a feasible solution a *fractional matching*; its cost upper-bounds $c(M)$

Proof of $5/3$ -approximation

- Want: a fractional matching y with $E[c(y)] \leq \frac{2}{3}c(x^*)$
 $x^* :=$ optimal Held-Karp solution

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$$\text{(Matching)} \quad \begin{cases} \sum_{e \in \delta(S)} y_e \geq 1, & \forall S \subset V, |S \cap T| \text{ odd} \\ y \in \mathbb{R}_+^E \end{cases}$$

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- Want: a fractional matching y with $E[c(y)] \leq \frac{2}{3}c(x^*)$
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- Take $y := \alpha \chi_{\mathcal{F}} + \beta x^*$ for $\alpha = \beta = \frac{1}{3}$

	$\chi_{\mathcal{F}}$	x^*	y
LB on T -odd s - t cut capacities		1	
LB on nonseparating cut capacities		2	

$$\text{(Held-Karp)} \quad \left\{ \begin{array}{l} \sum_{e \in \delta(\{s\})} x_e = \sum_{e \in \delta(\{t\})} x_e = 1 \\ \sum_{e \in \delta(\{v\})} x_e = 2, \quad \forall v \in V \setminus \{s, t\} \\ \sum_{e \in \delta(S)} x_e \geq 1, \quad \forall S \subsetneq V, \{|s, t\} \cap S| = 1 \\ \sum_{e \in \delta(S)} x_e \geq 2, \quad \forall S \subsetneq V, \{|s, t\} \cap S| \neq 1, S \neq \emptyset \\ 0 \leq x_e \leq 1 \quad \forall e \in E \end{array} \right.$$

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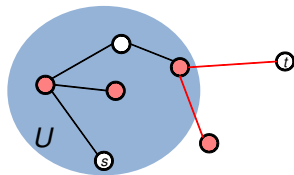
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LB on T -odd s - t cut capacities	2	1	
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Lemma

An s - t cut (U, \bar{U}) that is odd w.r.t. T (i.e., $|U \cap T|$ is odd) has at least two tree edges in it



Proof of 5/3-approximation

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	$\chi_{\mathcal{F}}$	x^*	y
LB on T -odd s - t cut capacities	2	1	$2\alpha + \beta = 1$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta = 1$

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- $E[c(y)] = \alpha E[c(\chi_{\mathcal{T}})] + \beta c(x^*) = (\alpha + \beta)c(x^*)$
- $E[c(H)] \leq E[c(\mathcal{T})] + E[c(M)] \leq (1 + \alpha + \beta)c(x^*)$

Theorem

The given algorithm is a $(1 + \alpha + \beta)$ -approximation algorithm

Improvement upon 5/3

	$\chi_{\mathcal{J}}$	x^*	y
LB on T -odd s - t cut capacities	2	1	$2\alpha + \beta$
LB on nonseparating cut capacities	1	2	$\alpha + 2\beta$

- Perturb α and β
 - In particular, decrease α by 2ϵ and increase β by ϵ

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- $E[c(y)] = (\alpha + \beta)c(x^*)$ decreases by $\epsilon c(x^*)$
- $\alpha + 2\beta$ unchanged; nonseparating cuts remain satisfied
- T -odd s - t cuts with small capacity may become violated
 - If violated, by at most $d := O(\epsilon)$

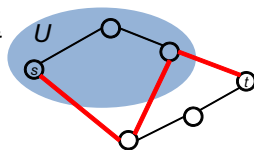
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Definition

For $0 < \tau \leq 1$, a τ -narrow cut (U, \bar{U}) is an s - t cut with $\sum_{e \in \delta(U)} x_e^* < 1 + \tau$



Improvement upon $5/3$

- \mathcal{T} -narrow cuts may be violated when they are T -odd

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Lemma

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Proof.

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- $\Pr[e \in \mathcal{T}] = x_e^*$

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- (U, \bar{U}) has at least one tree edge in it
- If (U, \bar{U}) is odd w.r.t. T , it must have another tree edge in it

□

- $\Pr[e \in \mathcal{T}] = x_e^*$

Lemma

An s - t cut (U, \bar{U}) that is odd w.r.t. T (i.e., $|U \cap T|$ is odd) has at least two tree edges in it

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- $y := \alpha \chi_{\mathcal{G}} + \beta x^* + r$
- For each e , if e is in a τ -narrow cut that is odd w.r.t. T , set $r_e := dx_e^*$

Claim y is a fractional matching

Claim $E[c(r)] \leq d_{\tau} c(x^*)$

Improvement upon $5/3$

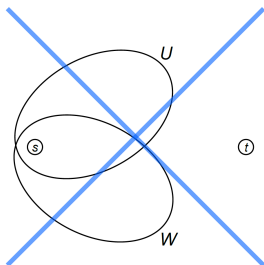
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- τ -narrow cuts are not disjoint, but “almost” disjoint

Lemma

τ -narrow cuts do not cross: i.e., for τ -narrow cuts (U, \bar{U}) and (W, \bar{W}) with $s \in U, W$, either $U \subset W$ or $W \subset U$.

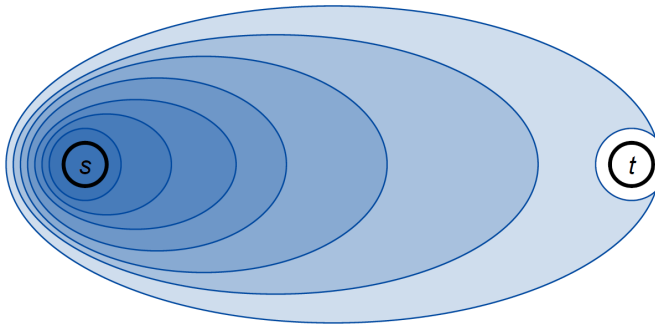


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Lemma

Each τ -narrow cut has a “representative” edge set of capacity $\geq 1 - \frac{\tau}{2}$, and they are mutually disjoint

The Main Result

Theorem

Best-of-many Christofides' algorithm is a deterministic ϕ -approximation algorithm for the s - t path TSP for the general metric, where $\phi = \frac{1+\sqrt{5}}{2} < 1.6181$ is the golden ratio

Open Questions

- Circuit TSP
 - Is there a better than $3/2$ -approximation algorithm?
 - Do our techniques extend to the circuit TSP?

Thank you.