Lecture 3: CountSketch, Graph sketching, ℓ_0 Samplers

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EPFL

May 26, 2017

- CountSketch (recap and proofs)
- Graph streaming
- Connectivity via sketching
- Designing ℓ_0 samplers

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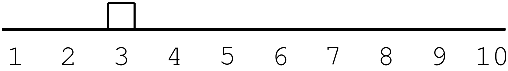

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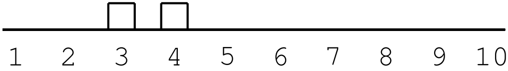

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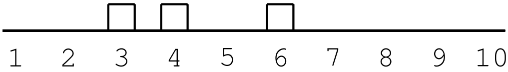

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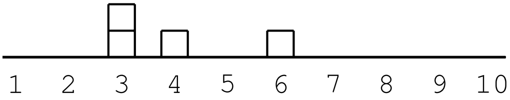

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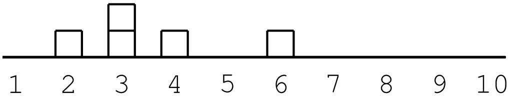

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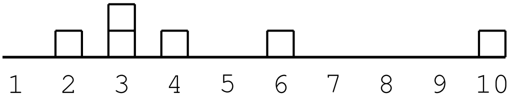

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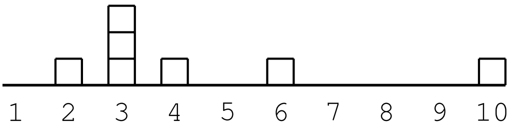

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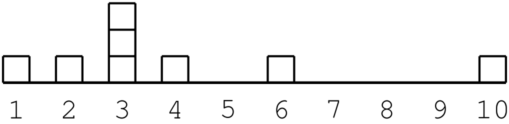

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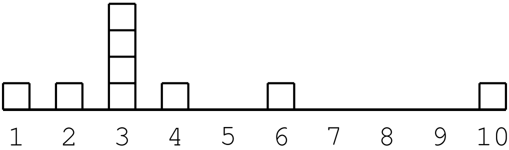

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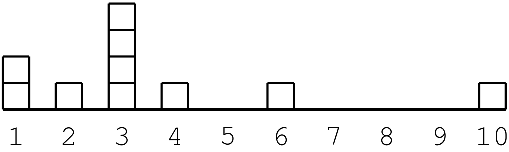

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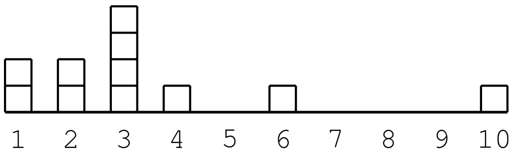

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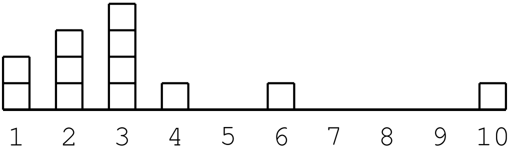

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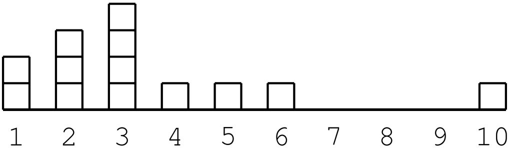

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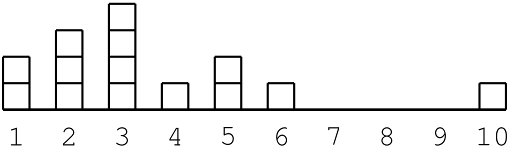

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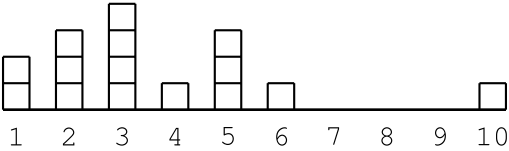

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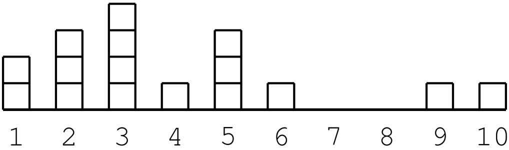

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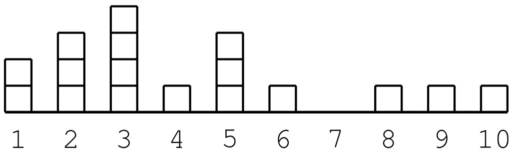

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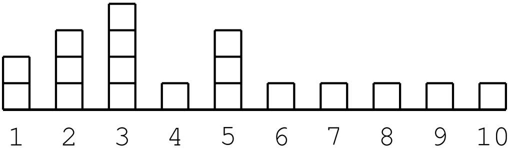

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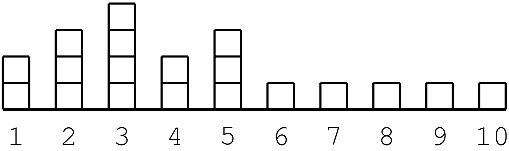

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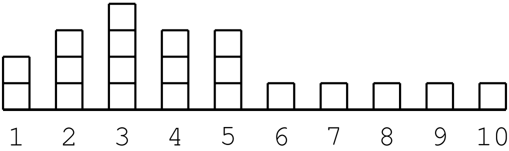

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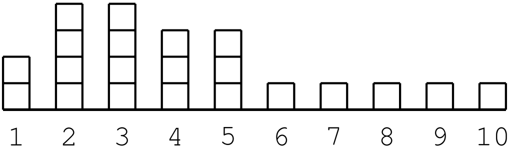

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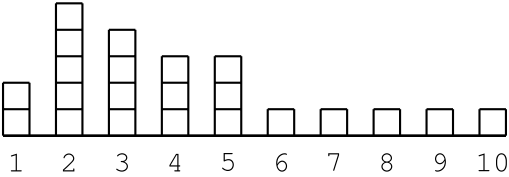

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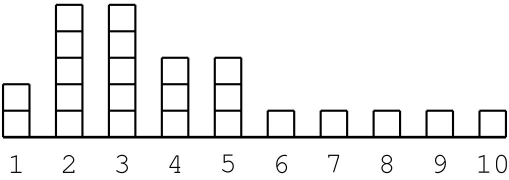

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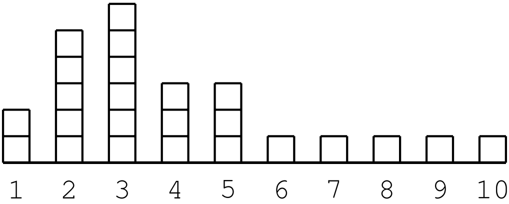

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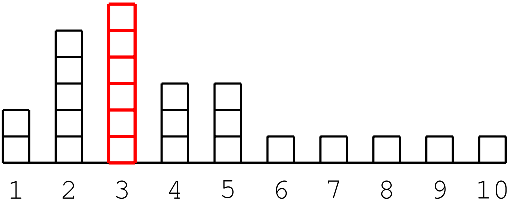

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Main primitive: ApproxPointQuery in small space

Observe a stream of updates, maintain small space data structure

Task: after observing the stream, given $i \in \{1, 2, \ldots, m\}$, compute estimate $\widehat{f_i}$ of f_i

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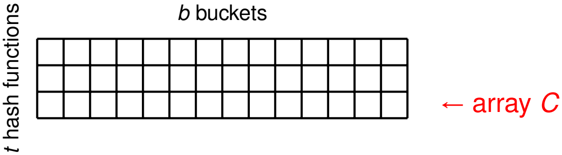
To be specified:

- space complexity?
- quality of approximation?
- success probability?

ApproxPointQuery

Choose

- t random hash functions $h_1, h_2, \dots, h_t$ from items $[m]$ to $b \approx k$ buckets $\{1, 2, \dots, b\}$
- t random hash functions $s_1, s_2, \dots, s_t$ from items $[m]$ to $\{-1, +1\}$



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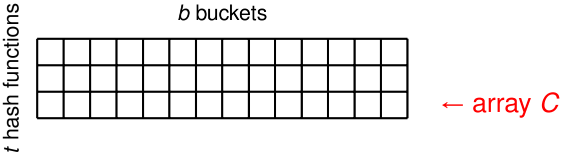

The algorithm runs t independent copies of basic estimate:

UPDATE(C, i)
for $r \in [1:t]$
  $C[r, h_r(i)] \leftarrow C[r, h_r(i)] + s_r(i)$
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ESTIMATE(C, i)
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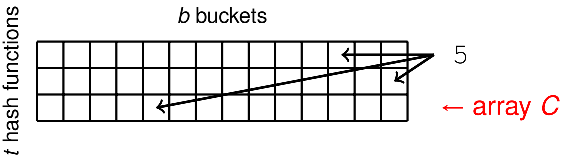

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$O(\log N)$ repetitions ensure estimates are correct **for all** i with high probability

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If $b \geq 8 \max\left\{k, \frac{32 \sum_{j \in TAIL} f_j^2}{(\varepsilon f_k)^2}\right\}$ *and* $t = O(\log N)$, *then for every* $i \in [m]$

$$|\mathit{median}_r \{C[r, h_r(i)] \cdot s_r(i)\} - f_i(p)| \leq \varepsilon f_k$$

at every point $p \in [1:N]$ *in the stream.*

($f_i(p)$ is the frequency of i up to position p)

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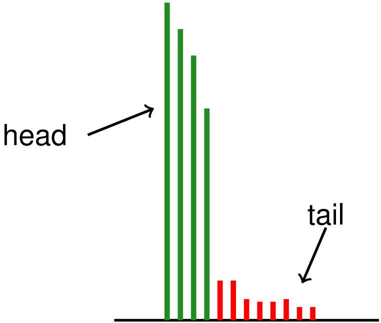
How large is b?

Space complexity

$$\text{Set } b = 8\max\left\{k, \frac{32\sum_{j\in TAIL} f_j^2}{(\varepsilon f_k)^2}\right\}$$

Note that $b = O(k/\varepsilon^2)$ if $\frac{1}{k}\sum_{j\in TAIL} f_j^2 = O(f_k^2)$



In practice, choose b subject to space constraints, detect elements with counts above $O\left(\varepsilon\sqrt{\frac{1}{k}\sum_{j\in TAIL} f_j^2}\right)$

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Set $k = 1$. Suppose that `1` appears $\sqrt{N}$ times in the stream, and other $N - \sqrt{N}$ elements are distinct

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Remarkable, as `1` appears only in $\sqrt{N}$ positions out of N: a vanishingly small fraction of positions!

Final algorithm: CountSketch

FindApproxTop(S, k, ε): returns set of k items such that $f_i \geq (1-\varepsilon) f_k$ for all returned i

(In fact also every i with $f_i \geq (1-\varepsilon) f_k$ is reported)

ApproxPointQuery(S, i, ε): returns $\widehat{f_i} \in [f_i - \varepsilon f_k, f_i + \varepsilon f_k]$

Find head items if they contribute the bulk of the stream in ℓ_2 sense

CountSketch: proof details

UPDATE(C, i)
for $r \in [1:t]$
$C[r, h_r(i)] \leftarrow C[r, h_r(i)] + s_r(i)$
end for

ESTIMATE(C, i)
return $\text{median}_r \{C[r, h_r(i)] \cdot s_r(i)\}$

By basic estimate analysis for every $r \in [1:t]$

$$\mathbf{E}[C[r, h_r(i)] \cdot s_r(i)] = f_i$$

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and

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Show that all three events hold simultaneously with probability strictly bigger than $1/2$ – so median gives good estimate

(No) collisions with head items

$\textsc{No-Collisions}_r(i)$:=event that

$$\{j \in HEAD \setminus i : h_r(j) = h_r(i)\} = \emptyset,$$

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Suppose that $b \geq 8k$. Then by the union bound

$$\begin{aligned}\mathbf{Pr}[\textsc{No-Collisions}_r(i)] &\geq 1 - k/b \\ &\geq 1 - 1/8\end{aligned}$$

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Small variance from tail elements

SMALL-VARIANCE$_r(i)$:=event that

$$\sum_{\substack{j\in TAIL, j\neq i\\ h_r(j)=h_r(i)}} f_j^2 \leq \frac{8}{b}\sum_{j\in TAIL} f_j^2$$

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For every $i, j \in [m], i \neq j$ and $r \in [1:t]$

$$\mathbf{Pr}_{h_r}[h_r(i) = h_r(j)] = 1/b \quad (b \text{ is the number of buckets})$$

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So by linearity of expectation

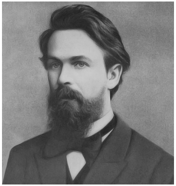
$$\begin{aligned}\mathbf{E}\left[\sum_{\substack{j\in TAIL, j\neq i\\ h_r(j)=h_r(i)}} f_j^2\right] &= \sum_{j\in TAIL, j\neq i} f_j^2 \cdot \mathbf{Pr}_{h_r}[h_r(i)=h_r(j)]\\ &\leq \frac{1}{b}\sum_{j\in TAIL} f_j^2\end{aligned}$$

Markov's inequality

Theorem

For every non-negative random variable X with mean $\mu \geq 0$, and every $k \geq 1$ one has

$$\mathbf{Pr}[X \geq k \cdot \mu] \leq 1/k$$



We proved that

$$\mathbf{E}\left[\sum_{\substack{j\in \mathit{TAIL}, j\neq i\\ h_r(j)=h_r(i)}} f_j^2\right] \leq \frac{1}{b}\sum_{j\in \mathit{TAIL}} f_j^2$$

By Markov's inequality one has, for every i and every r,

$$\mathbf{Pr}[\textsc{Small-Variance}_r(i)] \geq 1 - 1/8$$

$\textsc{No-Collisions}_r(i)$ and $\textsc{Small-Variance}_r(i)$: recap

Consider contribution of head and tail items separately:

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Conditioned on $\textsc{No-Collisions}_r(i)$ and $\textsc{Small-Variance}_r(i)$

- first term is zero
- second term is at most

$$\frac{8}{b} \sum_{j \in TAIL} f_j^2$$

Small deviation event

$\text{SMALL-DEVIATION}_r(i)$=event that

$$(C[r,h_r(i)]\cdot s_r(i)-f_i)^2\leq 8\mathbf{Var}(C[r,h_r(i)]\cdot s_r(i)).$$

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By Chebyshev's inequality one has, for every i and every r,

$$\mathbf{Pr}[\textsc{Small-Deviation}_r(i)] \geq 1 - 1/8$$

$$\mathbf{Pr}[\textsc{Small-Variance}_r(i)] \geq 1 - 1/8$$

$$\mathbf{Pr}[\textsc{No-Collisions}_r(i)] \geq 1 - 1/8$$

$$\mathbf{Pr}[\textsc{Small-Deviation}_r(i)] \geq 1 - 1/8$$

So by the union bound

$$\mathbf{Pr}[\textsc{Small-Variance}_r(i) \text{ and } \textsc{No-Collisions}_r(i) \\ \text{ and } \textsc{Small-Deviation}_r(i)] \geq 5/8.$$

Let

$$\gamma := \sqrt{\frac{1}{b}\sum_{j \in TAIL} f_j^2}$$

For every $p \in [1:N]$ let $f_i(p) :=$frequency of i up to position p

Lemma

If $b \geq 8k$, then for every i, every $r \in [1:t]$,

$$\mathbf{Pr}[|C[r, h_r(i)] \cdot s_r(i) - f_i| \leq 8\gamma] \geq 5/8$$

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For every $p \in [1:N]$ let $f_i(p)$:=frequency of i up to position p

Lemma

If $b \geq 8k$ and $t \geq A \log N$ for an absolute constant $A > 0$, then for every i, with probability $\geq 1 - 1/N^4$

$$\left| median_r \{C[r, h_r(i)] \cdot s_r(i)\} - f_i \right| \leq 8\gamma$$

at the end of the stream.

Proof.

Chernoff bounds. □

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Lemma

If $b \geq 8\max\left\{k, \frac{32\sum_{j\in TAIL} f_j^2}{(\varepsilon f_k)^2}\right\}$ *and* $t \geq A\log N$ *for an absolute constant* $A > 0$*, then with probability* $\geq 1 - 1/N^3$ *for every* $i \in [m]$

$$\left|median_r\{C[r, h_r(i)] \cdot s_r(i)\} - f_i(p)\right| \leq \varepsilon f_k$$

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Proof.

Substitute value of b into definition of γ:

$$\gamma = \sqrt{\frac{1}{b}\sum_{j\in TAIL} f_i^2} \leq \varepsilon f_k/8$$

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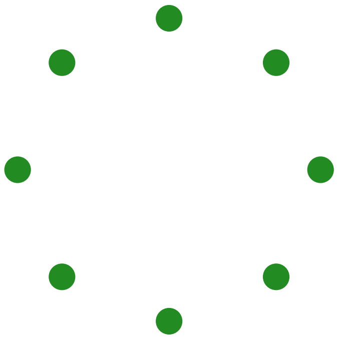
Streaming model

- streaming model: edges of G arrive in an arbitrary order in a stream;
- algorithm can only use $\widetilde{O}(n)$ space
- several passes over the stream



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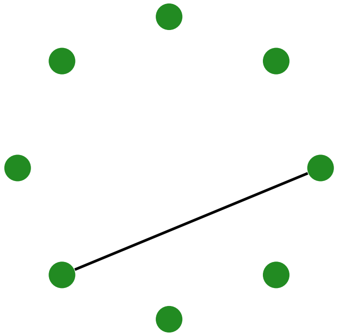

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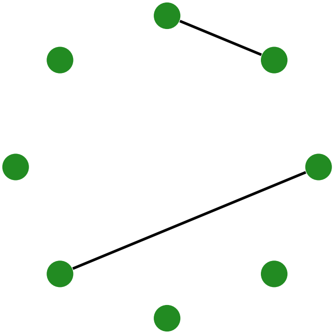

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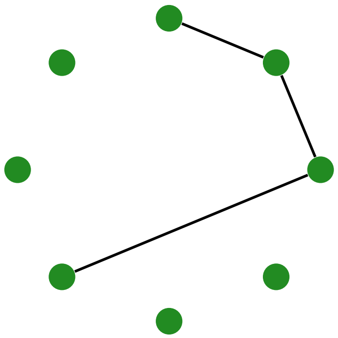

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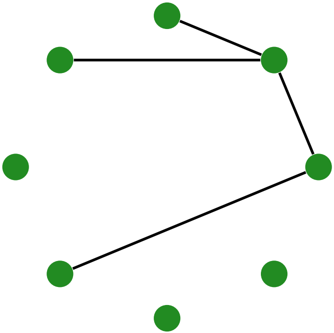

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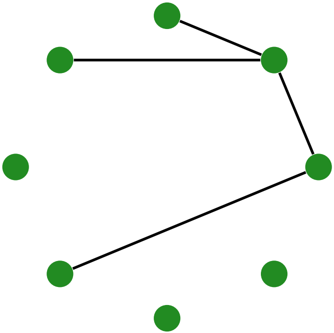

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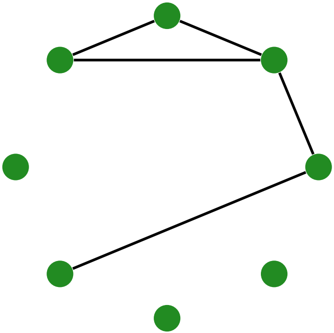

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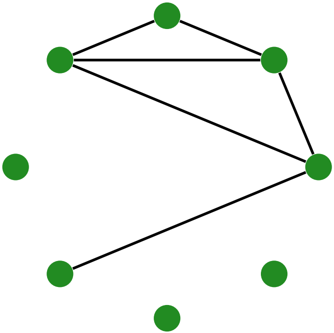

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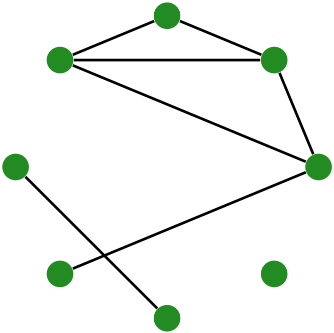

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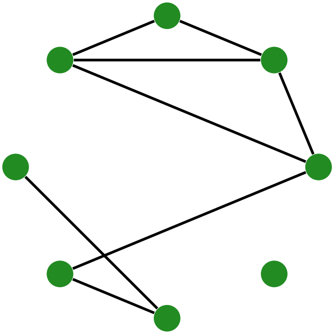

Streaming model

- streaming model: edges of G arrive in an arbitrary order in a stream;
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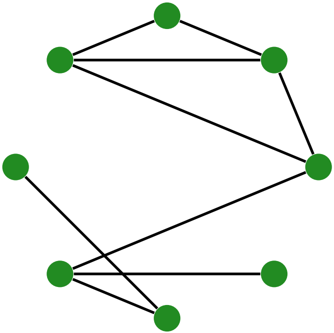

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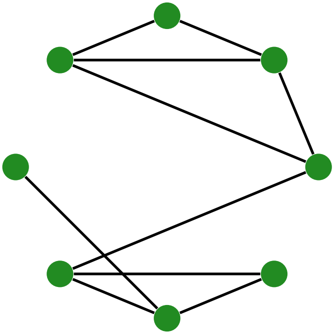

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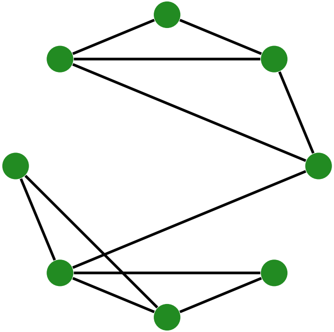

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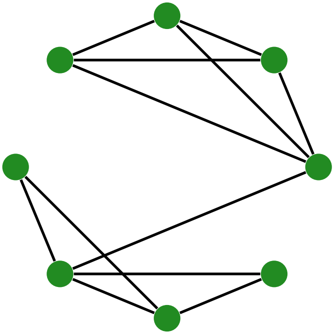

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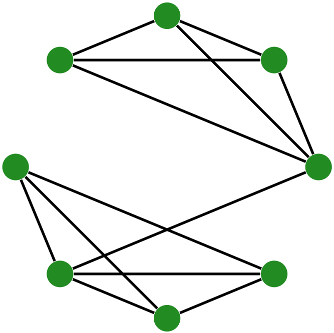

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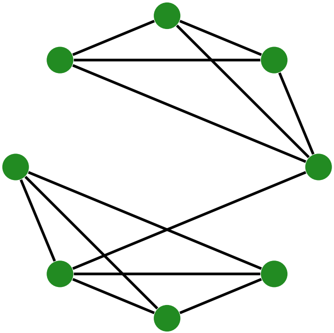

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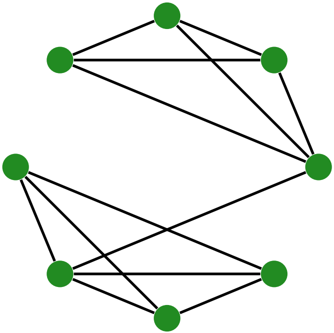

Insertion-only stream

Construct a spanning tree of the graph G in a single pass?

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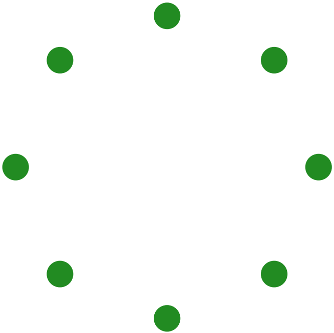
Spanning trees in insertion-only streams



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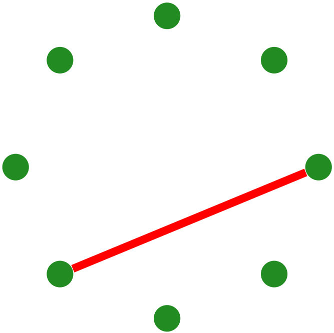

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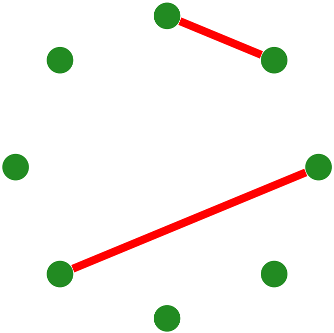

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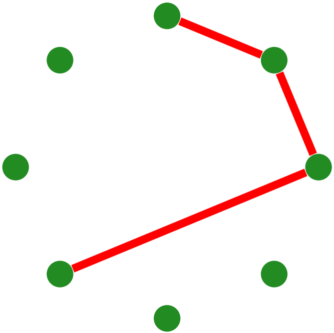

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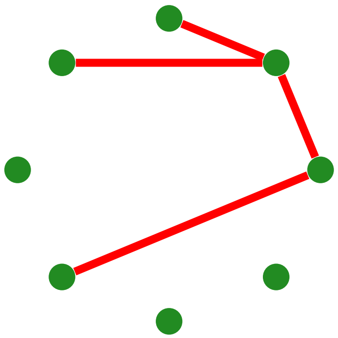

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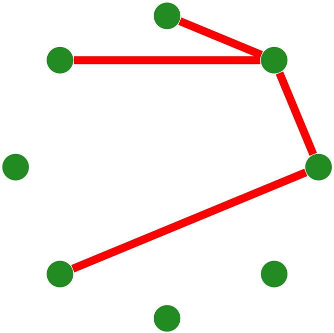

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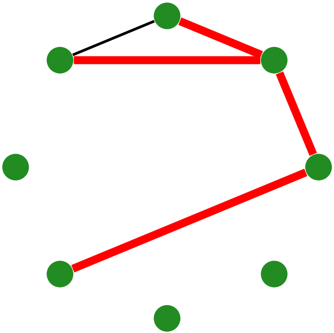

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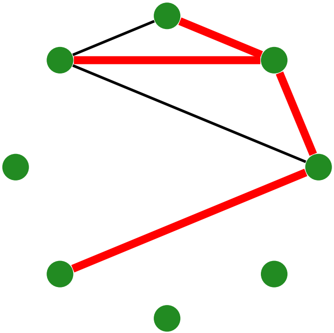

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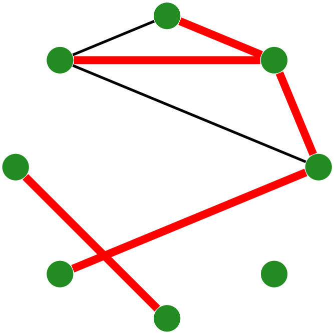

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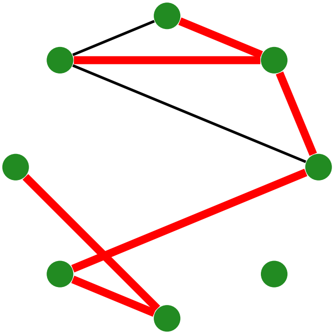

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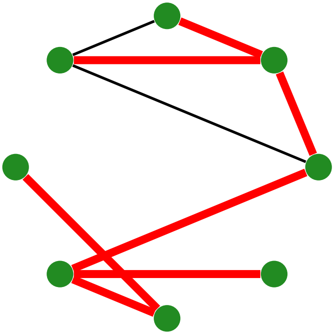

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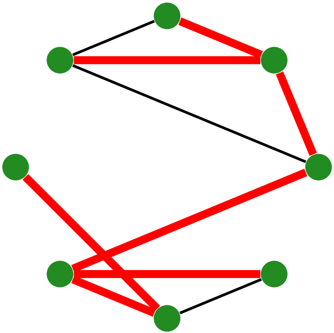

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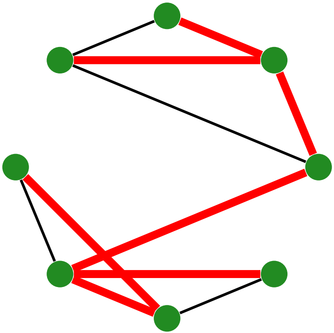

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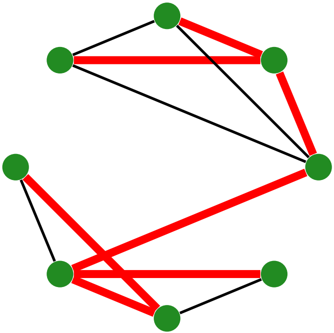

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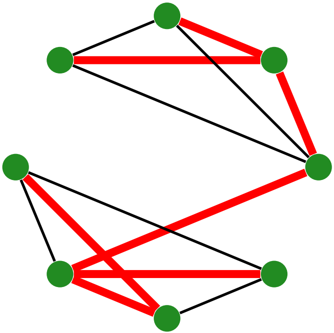

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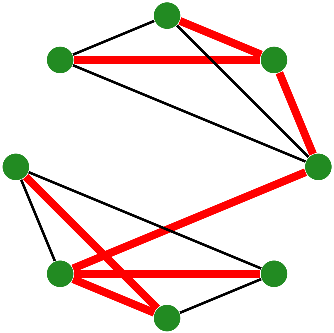

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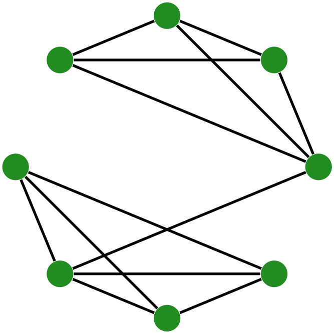
Many modern networks evolve over time, edges both inserted and deleted

Construct spanning trees in dynamic streams in small space?

What if we have deletions?



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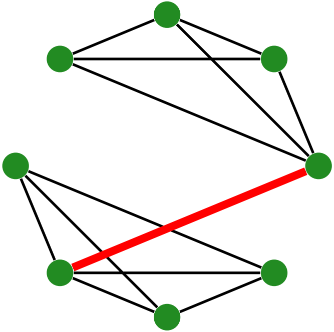

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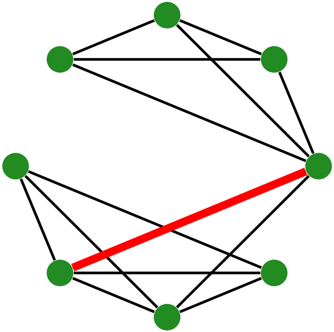

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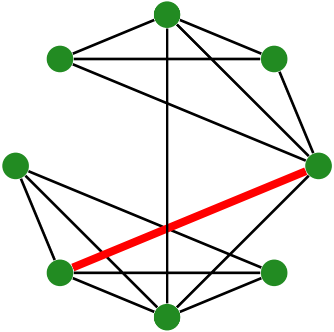

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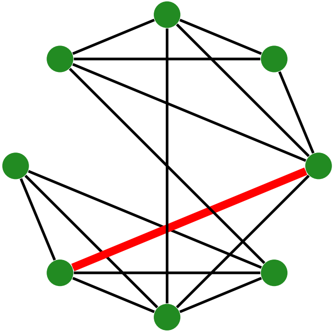

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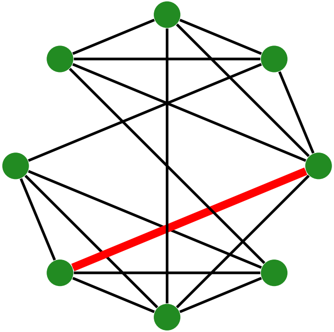

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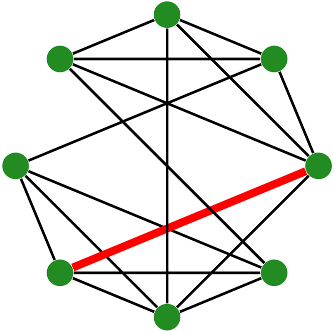

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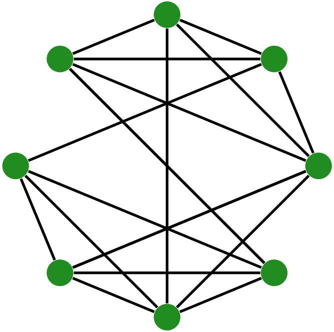

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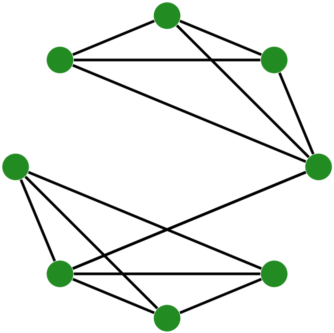

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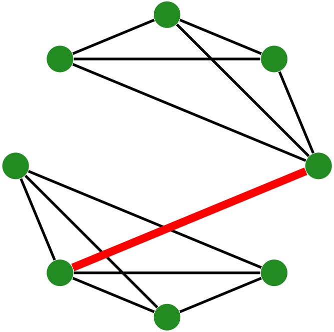
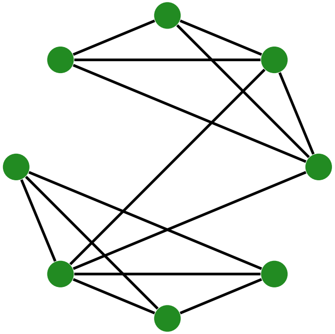


Very different algorithms are needed...

Graph sketching

Main idea: apply classical sketching techniques on the edge incidence matrix of a graph



$$B = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \end{bmatrix}$$

$\binom{n}{2} \times n$

Each row of B=potential edge in G.

If $e = (u, v) \in E$, then $b_e = \chi_u - \chi_v$, otherwise $b_e = 0$

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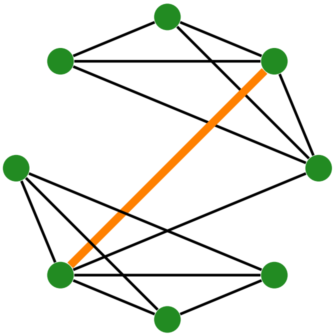

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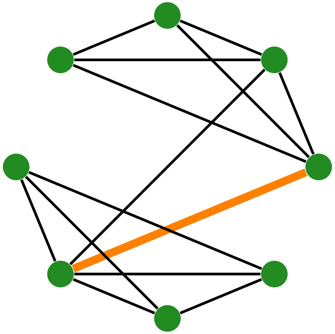

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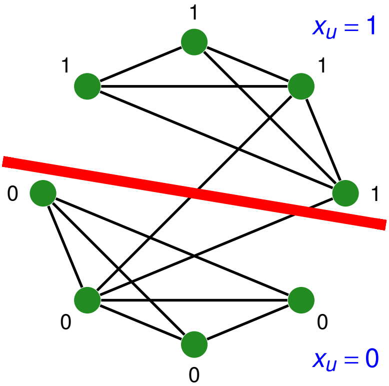
Graph sketching

For every $S \subseteq V$ let

$$\delta(S) := E \cap (S \times (V \setminus S))$$

denote the edges crossing the cut. Let $x = \mathbf{1}_S$ (indicator of S).

$$\boxed{Bx \text{ is the (signed) indicator of } \delta(S)}$$



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- Graph streaming
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- Designing ℓ_0 samplers

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- **Connectivity via sketching**
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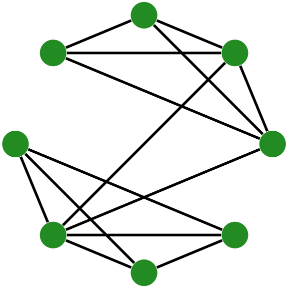
A simple algorithm for connectivity



$F^0 \leftarrow \emptyset$

$C^0 \leftarrow V$ $\quad \triangleright$ current connected components

For $t = 0$ **to** T $\quad \triangleright$ $T = O(\log n)$

  For each $u \in C^t$

    Choose an edge in $\delta(u)$ (nonzero of $B \cdot \mathbf{1}_u$)

  End For

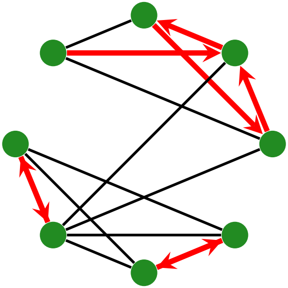
  $F^{t+1} \leftarrow F^t \cup \{$spanning forest on selected edges$\}$

  $C^{t+1} \leftarrow \{$new connected components$\}$

End

return F^{T+1}

A simple algorithm for connectivity



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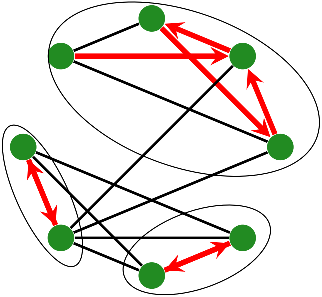

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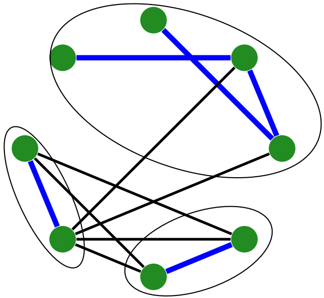

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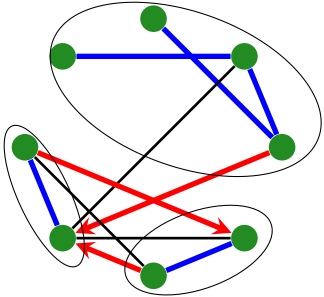

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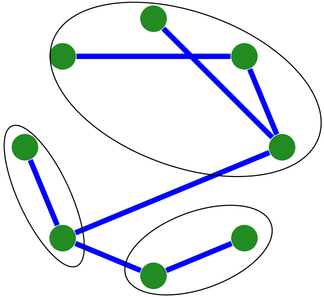

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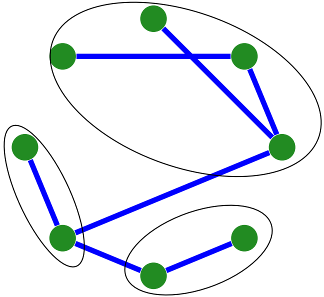

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(Sample nnz of $B \cdot \mathbf{1}_u$?)

ℓ_0-samplers

Definition

A δ-error ℓ_0 sampler is

- a linear sketch $S \in \mathbb{R}^{m \times n}$
- a decoding primitive $Dec : \mathbb{R}^m \to [n]$

such that for every $x \in \mathbb{R}^n$ with integer entries $J \leftarrow Dec(Sx)$ satisfies

$$||J - UNIF_{\mathrm{supp}(x)}||_{TVD} \leq \delta.$$

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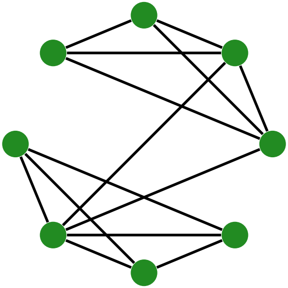
Recent constructions of ℓ_p samplers due to

Frahling-Indyk-Sohler'08, Andoni-Krauthgamer-Onak'11, Jowhari-Saglam-Tardos'11, Nelson-Pachocki-Wang'17, Kapralov-Nelson-Pachocki-Wang-Woodruff-Yahyazadeh'17

Connectivity via sketching (Ahn-Guha-McGregor'12)



$F^0 \leftarrow \emptyset$

$C^0 \leftarrow V$ $\quad\triangleright$ current connected components

For $t = 0$ **to** T $\quad\triangleright$ $T = O(\log n)$

  For each $u \in C^t$

    Choose an edge in $\delta(u)$

  End For

 $F^{t+1} \leftarrow F^t \cup$ {spanning forest on selected edges}

 $C^{t+1} \leftarrow$ {new connected components}

End

return F^{T+1}

$S^1, \ldots, S^T \leftarrow \ell_0$-samplers

Maintain $S^1 B, \ldots, S^T B$

Run $Dec(S^t B \cdot \mathbf{1}_u)$

Some remarks

Why did we need T sketches $S^1, \ldots, S^T$?

Very surprising: decoding is adaptive ($T = O(\log n)$ rounds), but sketch is not

Which other graph problems admit sketching solutions?

- CountSketch (recap and proofs)
- Graph streaming
- Connectivity via sketching
- Designing ℓ_0 samplers

- CountSketch (recap and proofs)
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- **Designing ℓ_0 samplers**

ℓ_0-samplers

Definition

A δ-error ℓ_0 sampler is

- a linear sketch $S \in \mathbb{R}^{m \times n}$
- a decoding primitive $Dec : \mathbb{R}^m \to [n]$

such that for every $x \in \mathbb{R}^n$ with integer entries $J \leftarrow Dec(Sx)$ satisfies

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Recent constructions of ℓ_p samplers due to

Frahling-Indyk-Sohler'08, Andoni-Krauthgamer-Onak'11, Jowhari-Saglam-Tardos'11, Nelson-Pachocki-Wang'17, Kapralov-Nelson-Pachocki-Wang-Woodruff-Yahyazadeh'17

ℓ_p sampling problem

- Single pass over the data: $i_1, i_2, \ldots, i_N$

  Assume N is known

- Output item i with probability $\sim f_i^p$

  (f_i=number of occurrences of i)

- Small storage: will get $\log^{O(1)} N$

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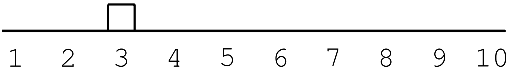

3

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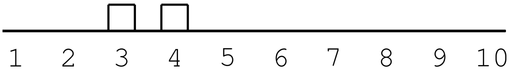

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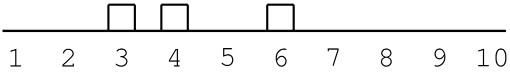

3 4 6

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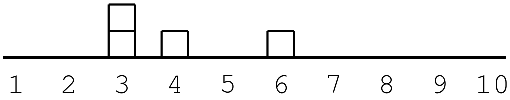

3 4 6 3

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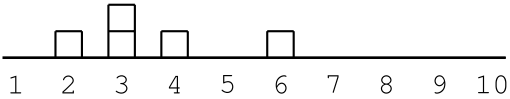

3 4 6 3 2

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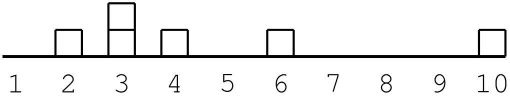

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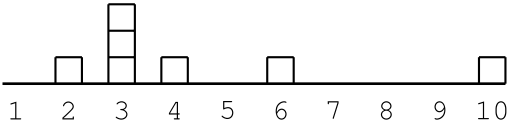

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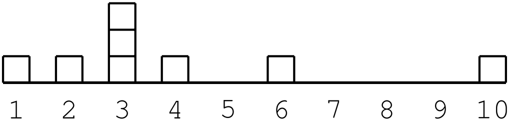

3 4 6 3 2 10 3 1

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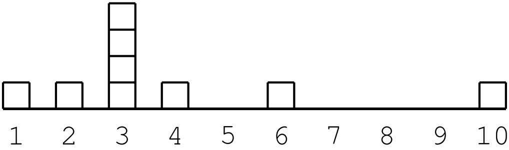

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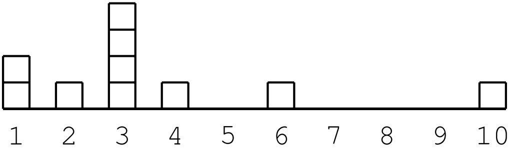

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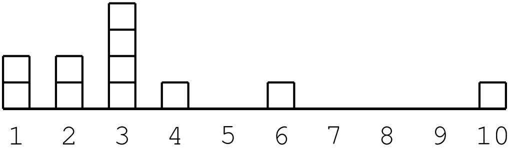

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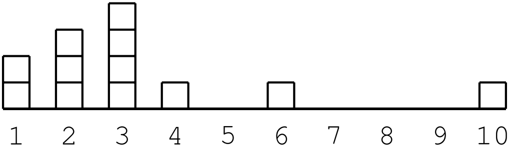

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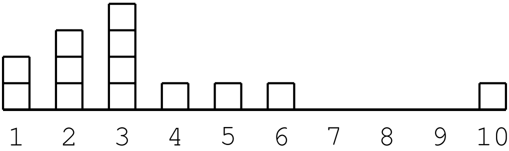

3 4 6 3 2 10 3 1 3 1 2 2 5

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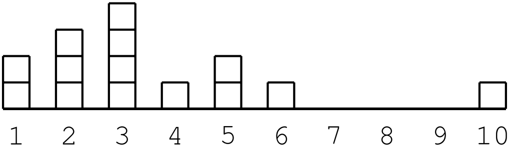

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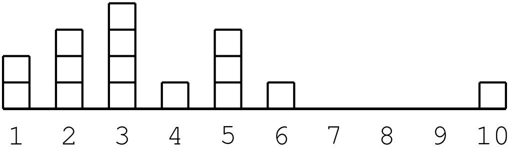

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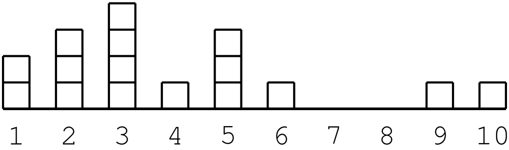

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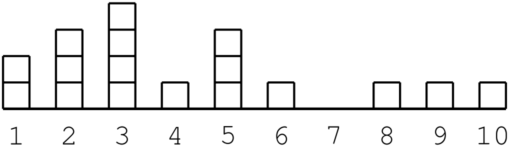

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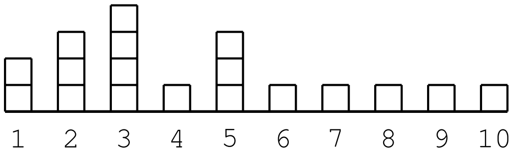

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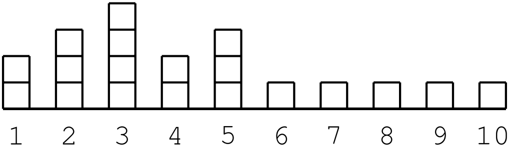

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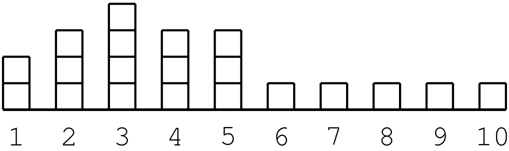

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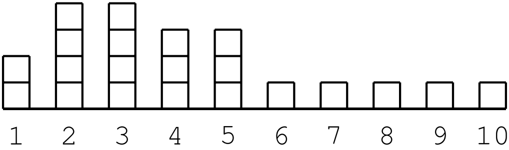

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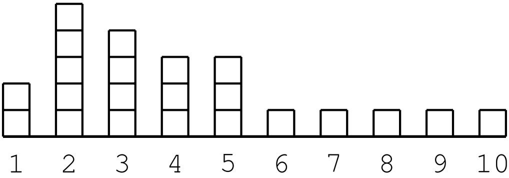

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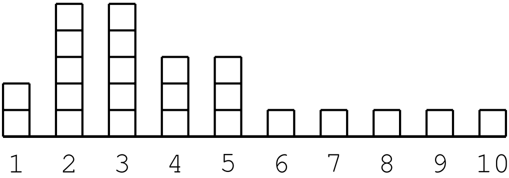

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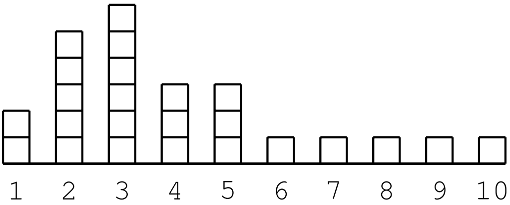

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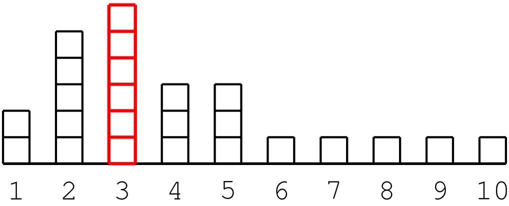

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ℓ_0 sampler construction (sketch)

Main idea:

- if x is 1-sparse (has a single nonzero), can recover x using few rows

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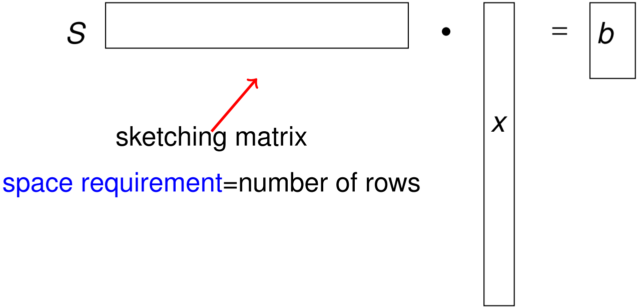
- if x is 1-sparse (has a single nonzero), can recover x using few rows
- if x is t-sparse, a subsampling of x at rate $\approx 1/t$ is likely 1-sparse

Recovering 1-sparse signals

Design a sketch S from which any $x \in \mathbb{R}^n$ with $\mathrm{supp}(x) = 1$ can be recovered with probability 1?



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$$S \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots & n-1 & n \end{bmatrix} \bullet \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

sketching matrix

space requirement=number of rows

Suppose x has one nonzero: $x = \alpha \cdot \mathbf{e}_{i^*}$

Compute

$$A = \sum_{i=1}^{n} x_i \quad = \langle u, x \rangle = \alpha$$

$$B = \sum_{i=1}^{n} i \cdot x_i = \langle v, x \rangle = \alpha \cdot i^*$$

So

$$\alpha = \langle x, u \rangle$$

and

$$i^* = \frac{\langle x, v \rangle}{\langle x, u \rangle}$$

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Try $O(\log n)$ powers of 2, run 1-sparse recovery on x_S!

Also need to verify that recovery was successful (can be done)

Optimal bounds for ℓ_0-samplers

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Jowhari-Saglam-Tardos'11: there exist δ-error ℓ_0-samplers with $m = O(\log n \log(1/\delta))$ rows.

Kapralov-Nelson-Pachocki-Wang-Woodruff-Yahyazadeh'17: this space bound is optimal for $\delta > 2^{-n^{0.99}}$ (and more results)