Sparse Fourier Transform
(lecture 2)

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Given $x \in \mathbb{C}^n$, compute the Discrete Fourier Transform of $x$:

$$\hat{x}_f = \frac{1}{n} \sum_{j \in [n]} x_j \omega^{-f\cdot j},$$

where $\omega = e^{2\pi i / n}$ is the $n$-th root of unity.
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where $\omega = e^{2\pi i / n}$ is the $n$-th root of unity.

**Goal:** find the top $k$ coefficients of $\hat{x}$ approximately

In last lecture:

- 1-sparse noiseless case: two-point sampling
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- 1-sparse noisy case: $O(\log n \log \log \log n)$ time and samples
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In last lecture:

- 1-sparse noiseless case: two-point sampling
- 1-sparse noisy case: $O(\log n \log \log \log n)$ time and samples
- reduction from $k$-sparse to 1-sparse case, via filtering
Partition frequency domain into $B \approx k$ buckets

For each $j = 0, \ldots, B - 1$ let

$$\hat{u}_j f = \begin{cases} 
\hat{x}_f, & \text{if } f \in j\text{-th bucket} \\
0, & \text{o.w.}
\end{cases}$$

Restricted to a bucket, signal is likely approximately 1-sparse!
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Restricted to a bucket, signal is likely approximately 1-sparse!
We want time domain access to $u^0$: for any $a = 0, \ldots, n-1$, compute

$$u_a^0 = \sum_{-\frac{n}{2B} \leq f \leq \frac{n}{2B}} \hat{\chi}_f \cdot \omega^{f \cdot a}.$$ 

Let

$$\hat{G}_f = \begin{cases} 1, & \text{if } f \in \left[ -\frac{n}{2B} : \frac{n}{2B} \right] \\ 0 & \text{o.w.} \end{cases}$$

Then

$$u_a^0 = (\hat{x} + a \ast \hat{G})(0)$$
We want time domain access to $u^0$: for any $a = 0, \ldots, n-1$, compute

$$u^0_a = \sum_{-\frac{n}{2B} \leq f \leq \frac{n}{2B}} \hat{\chi}_f \cdot \omega^{f \cdot a}.$$

Let

$$\hat{G}_f = \begin{cases} 1, & \text{if } f \in \left[ -\frac{n}{2B}, \frac{n}{2B} \right] \\ 0 & \text{o.w.} \end{cases}$$

Then

$$u^0_a = (\hat{x}_{+a} * \hat{G})(0)$$

For any $j = 0, \ldots, B-1$

$$u^j_a = (\hat{x}_{+a} * \hat{G})(j \cdot \frac{n}{B})$$
Reducing $k$-sparse recovery to $1$-sparse recovery

For any $j = 0, \ldots, B - 1$

$$u_j^a = (\hat{x} + a \ast \hat{G})(j \cdot \frac{n}{B})$$
Reducing $k$-sparse recovery to 1-sparse recovery

For any $j = 0, \ldots, B - 1$

$$u_{ja}^j = (\hat{x} + a \ast \hat{G})(j \cdot \frac{n}{B})$$
Reducing $k$-sparse recovery to 1-sparse recovery

For any $j = 0, \ldots, B - 1$

$$u^j_a = (\widehat{x} + a \ast \widehat{G})(j \cdot \frac{n}{B})$$
Need to evaluate

\[(\hat{x} + a \ast \hat{G})(j \cdot \frac{n}{B})\]

for \(j = 0, \ldots, B - 1\).

We have access to \(x\), not \(\hat{x}\)...

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We have access to \(x\), not \(\hat{x}\).

By the convolution identity

\[\hat{\chi} + a \ast \hat{G} = (\overline{x + a} \cdot \overline{G})\]
Need to evaluate

\[(\hat{x} + a \ast \hat{G})(j \cdot \frac{n}{B})\]

for \(j = 0, \ldots, B - 1\).

We have access to \(x\), not \(\hat{x}\)...

By the convolution identity

\[\hat{x} + a \ast \hat{G} = (\overline{x + a} \ast G)\]

Suffices to compute

\[\overline{x + a} \cdot G_j \cdot \frac{n}{B}, j = 0, \ldots, B - 1\]
Suffices to compute

\[ x \cdot a \cdot G_{j_0} \cdot n \cdot B, j = 0, \ldots, B - 1 \]
Suffices to compute

\[ \hat{x} \cdot G_{j\cdot \frac{n}{B}}, j = 0, \ldots, B - 1 \]
Suffices to compute

\[ \hat{x} \cdot \hat{G}_{j,n} \cdot B, j = 0, \ldots, B - 1 \]

Sample complexity? Runtime?
Suffices to compute

$$\hat{x} \cdot \hat{G}_{j,n} B^j, j = 0, \ldots, B - 1$$

Sample complexity? Runtime?
To sample all signals $u^j, j = 0, \ldots, B - 1$ in time domain, it suffices to compute 

$$\widehat{x \cdot G_j \cdot n}^\frac{j}{B}, j = 0, \ldots, B - 1$$

Computing $x \cdot G$ takes $\text{supp}(G)$ samples.

Design $G$ with $\text{supp}(G) \approx k$ that approximates rectangular filter?
In this lecture:

- permuting frequencies
- filter construction
1. Pseudorandom spectrum permutations
2. Filter construction
1. Pseudorandom spectrum permutations

2. Filter construction
Pseudorandom spectrum permutations

Permutation in time domain plus phase shift $\rightarrow$ permutation in frequency domain

Claim

Let $\sigma, b \in \mathbb{Z}_n$, $\sigma$ invertible modulo $n$. Let $y_j = x_{\sigma j} \omega^{-jb}$. Then $\hat{y}_f = \hat{x}_{\sigma^{-1}(f+b)}$. (proof on next slide; a close relative of time shift theorem)

Pseudorandom permutation:

- Select $b$ uniformly at random from $\mathbb{Z}_n$
- Select $\sigma$ uniformly at random from $\{1, 3, 5, \ldots, n-1\}$ (invertible numbers modulo $n$)
Pseudorandom spectrum permutations

Permutation in time domain plus phase shift $\implies$ permutation in frequency domain

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(proof on next slide; a close relative of time shift theorem)

Pseudorandom permutation:

- select $b$ uniformly at random from $[n]$
- select $\sigma$ uniformly at random from $\{1, 3, 5, \ldots, n-1\}$
  (invertible numbers modulo $n$)
Pseudorandom spectrum permutations

Claim

Let \( y_j = x_{\sigma j} \omega^{-jb} \). Then \( \hat{y}_f = \hat{x}_{\sigma^{-1}}(f+b) \).

Proof.

\[
\hat{y}_f = \frac{1}{n} \sum_{j \in [n]} y_j \omega^{-f \cdot j}
\]

\[
= \frac{1}{n} \sum_{j \in [n]} x_{\sigma j} \omega^{-(f+b) \cdot j}
\]

\[
= \frac{1}{n} \sum_{i \in [n]} x_{i} \omega^{-(f+b) \cdot \sigma^{-1} i} \quad \text{(change of variables } i = \sigma j)\]

\[
= \frac{1}{n} \sum_{i \in [n]} x_{i} \omega^{-\sigma^{-1}(f+b) \cdot i}
\]

\[
= \hat{x}_{\sigma^{-1}}(f+b)
\]
Design $G$ with $\text{supp}(G) \approx k$ that approximates rectangular filter?

Our filter $\hat{G}$ will approximate the boxcar. Bound collision probability now.
Partition frequency domain into buckets, permute spectrum
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Frequency domain

\[ \text{Frequency domain} \]

- Frequency \( i \) collides with frequency \( j \) only if \( |\sigma_i - \sigma_j| \leq n_B \).
Partition frequency domain into buckets, permute spectrum

Frequency $i$ collides with frequency $j$ only if $|\sigma_i - \sigma_j| \leq \frac{n}{B}$. 
Partition frequency domain into buckets, permute spectrum

Frequency $i$ collides with frequency $j$ only if $|\sigma_i - \sigma_j| \leq \frac{n}{B}$.
Collision probability

Lemma

Let $\sigma$ be a uniformly random odd number in $1, 2, \ldots, n$. Then for any $i, j \in [n], i \neq j$ one has

$$\text{Pr}_\sigma \left[ |\sigma \cdot i - \sigma j| \leq \frac{n}{B} \right] = O\left(\frac{1}{B}\right)$$
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$$\Pr_\sigma \left[ |\sigma \cdot i - \sigma j| \leq \frac{n}{B} \right] = O\left(\frac{1}{B}\right)$$

Proof.
Let $\Delta := i - j = d2^s$ for some odd $d$.

The orbit of $\sigma \cdot \Delta$ is $2^s \cdot d'$ for all odd $d'$.

There are $O\left(\frac{n}{B2^s}\right)$ values of $d'$ that make $\sigma \cdot \Delta$ fall into $[-\frac{n}{B}, \frac{n}{B}]$, out of $n/2^{s+1}$.

\[\text{Diagram:}\]

- Red dots represent the values of $\sigma i - \frac{n}{B}$, $\sigma i$, and $\sigma i + \frac{n}{B}$.
- The black dot marks the middle value.

\[\text{Diagram:}\]
Collision probability

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\Pr_{\sigma} \left[ |\sigma \cdot i - \sigma j| \leq \frac{n}{B} \right] = O\left( \frac{1}{B} \right)
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Proof.

Let \( \Delta := i - j = d2^s \) for some odd \( d \).

The orbit of \( \sigma \cdot \Delta \) is \( 2^s \cdot d' \) for all odd \( d' \).

There are \( O\left( \frac{n}{B2^s} \right) \) values of \( d' \) that make \( \sigma \cdot \Delta \) fall into \( [-\frac{n}{B}, \frac{n}{B}] \), out of \( n/2^{s+1} \).\]
1. Pseudorandom spectrum permutations
2. Filter construction
Rectangular buckets $\hat{G}$ have full support in time domain...

Approximate rectangular filter with a filter $G$ with small support?

Need $\text{supp}(G) \approx k$, so perhaps turn the filter around?
Let 

\[
G_j := \begin{cases} 
\frac{1}{B+1} & \text{if } j \in [-B/2, B/2] \\
0 & \text{otherwise}
\end{cases}
\]

Have \( \text{supp}(G) = B \approx k \), but \textit{buckets leak}.
In what follows: reduce leakage at the expense of increasing $\text{supp}(G)$
Window functions

Definition
A symmetric filter $G$ is a $(B, \delta)$-standard window function if

1. $\hat{G}_0 = 1$
2. $\hat{G}_f \geq 0$
3. $|\hat{G}_f| \leq \delta$ for $f \not\in \left[-\frac{n}{2B}, \frac{n}{2B}\right]$

ideal bucket

leakage to other buckets

bounded by $\delta \ll 1$
Window functions

Start with the sinc function:

\[ \hat{G}_f := \frac{\sin(\pi(B+1)f/n)}{(B+1) \cdot \pi f/n} \]
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\[ \hat{G}_f := \frac{\sin(\pi(B+1)f/n)}{(B+1) \cdot \pi f/n} \]

For all \( |f| > \frac{n}{2B} \) we have

\[ |\hat{G}_f| \leq \frac{1}{(B+1)\pi f/n} \leq \frac{1}{\pi/2} \leq 2/\pi \leq 0.9 \]
Consider powers of the sinc function:

\[ \hat{G}_f^r := \left( \frac{\sin(\pi(B+1)f/n)}{(B+1) \cdot \pi f/n} \right)^r \]

For all \( |f| > \frac{n}{2B} \) we have

\[ |\hat{G}_f|^r \leq (0.9)^r \]
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For all $|f| > \frac{n}{2B}$ we have

$$|\hat{G}_f|^r \leq (0.9)^r$$

So setting $r = O(\log(1/\delta))$ is sufficient!
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ideal bucket

leakage to other buckets bounded by $\delta \ll 1$

How large is $\text{supp}(G) \subseteq [-T, T]$?
Let \[ G_j := \begin{cases} \frac{1}{(B + 1)} & \text{if } j \in [-B/2, B/2] \\ 0 & \text{o.w.} \end{cases} \]

Let \( \hat{G}^r := (\hat{G}^0)^r \). How large is the support of \( G^r \)?
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Let \( \hat{G}^r := (\hat{G}^0)^r \). How large is the support of \( G^r \)?

By the convolution identity \( G^r = G^0 \ast G^0 \ast \ldots \ast G^0 \)
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Let \( \hat{G}^r := (\hat{G}^0)^r \). How large is the support of \( G^r \)?

By the convolution identity \( G^r = G^0 \ast G^0 \ast \ldots \ast G^0 \)

Support of \( G^0 \) is in \([ -B/2, B/2 ] \), so

\[ \text{supp}(G \ast \ldots \ast G) \subseteq [ -r \cdot B/2, r \cdot B/2 ] \]
Let \( G_j := \begin{cases} 1/(B+1) & \text{if } j \in [-B/2, B/2] \\ 0 & \text{o.w.} \end{cases} \)

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By the convolution identity \( G^r = G^0 \ast G^0 \ast \ldots \ast G^0 \)

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Let \( \hat{G}^r := (\hat{G}^0)^r \). How large is the support of \( G^r \)?

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G_j := \begin{cases} 
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0 & \text{o.w.} 
\end{cases}
\]

Let \( \hat{G}' := (\hat{G})^r \). How large is the support of \( G' \)?

By the convolution identity \( G' = G^0 \ast G^0 \ast \ldots \ast G^0 \)

Support of \( G^0 \) is in \([-B/2, B/2]\), so

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\text{supp}(G \ast \ldots \ast G) \subseteq [-r \cdot B/2, r \cdot B/2]
\]
Let $G_j := \begin{cases} 
1/(B+1) & \text{if } j \in [-B/2, B/2] \\
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\end{cases}$

Let $\hat{G}' := (\hat{G}^0)'$. How large is the support of $G'$?

By the convolution identity $G' = G^0 \ast G^0 \ast \ldots \ast G^0$

Support of $G^0$ is in $[-B/2, B/2]$, so

$$\text{supp}(G \ast \ldots \ast G) \subseteq [-r \cdot B/2, r \cdot B/2]$$
Let

\[ G_j := \begin{cases} 
  \frac{1}{B+1} & \text{if } j \in [-B/2, B/2] \\
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\end{cases} \]

Let \( \hat{G}' := (\hat{G}^0)^r \). How large is the support of \( G' \)?

By the convolution identity

\[ G' = G^0 * G^0 * \ldots * G^0 \]

Support of \( G^0 \) is in \([-B/2, B/2]\), so

\[ \text{supp}(G * \ldots * G) \subseteq [-r \cdot B/2, r \cdot B/2] \]
Let

\[ G_j := \begin{cases} 
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Let \( \hat{G}^r := (\hat{G}^0)^r \). How large is the support of \( G^r \)?

By the convolution identity \( G^r = G^0 \ast G^0 \ast \ldots \ast G^0 \)

Support of \( G^0 \) is in \([-B/2, B/2]\), so

\[ \text{supp}(G \ast \ldots \ast G) \subseteq [-r \cdot B/2, r \cdot B/2] \]
Flat window function

Definition
A symmetric filter $G$ is a $(B, \delta, \gamma)$-flat window function if
1. $\hat{G}_j \geq 1 - \delta$ for all $j \in \left[ -(1 - \gamma) \frac{n}{2B}, (1 - \gamma) \frac{n}{2B} \right]$
2. $\hat{G}_j \in [0, 1]$ for all $j$
3. $|\hat{G}_f| \leq \delta$ for $f \not\in \left[ -\frac{n}{2B}, \frac{n}{2B} \right]$
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$1 - \gamma$ fraction of bucket

ideal bucket

$\frac{-n}{2B}$ 0 $\frac{n}{2B}$
Flat window function

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2. $\hat{G}_j \in [0, 1]$ for all $j$
3. $|\hat{G}_f| \leq \delta$ for $f \not\in \left[-\frac{n}{2B}, \frac{n}{2B}\right]$
Let $H$ be a $(2B/\gamma, \delta/n)$-standard window function. Note that

$$|\hat{H}_f| \leq \delta/n$$

for all $f$ outside of

$$\left[-\gamma \frac{n}{4B}, \gamma \frac{n}{4B}\right].$$
Let $H$ be a $(2B/\gamma, \delta/n)$-standard window function. Note that

$$|\hat{H}_f| \leq \delta/n$$

for all $f$ outside of

$$\left[-\gamma \frac{n}{4B}, \gamma \frac{n}{4B}\right].$$
Flat window function – construction

Let $H$ be a $(2B/\gamma, \delta/n)$-standard window function. Note that

$$|\hat{H}_f| \leq \frac{\delta}{n}$$

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$$|\hat{H}_f| \leq \frac{\delta}{n}$$

for all $f$ outside of

$$\left[-\gamma \frac{n}{4B}, \gamma \frac{n}{4B}\right].$$
Flat window function – construction

To construct $\hat{G}$:

1. sum up shifts $\hat{H}_{-\Delta}$ over all $\Delta \in [-U, U]$, where
   \[ U = (1 - \gamma/2) \frac{n}{2B} \]

2. normalize so that $\hat{G}_0 = 1 \pm \delta$
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   \[
   U = (1 - \gamma/2) \frac{n}{2B}
   \]

2. normalize so that $\hat{G}_0 = 1 \pm \delta$

\[
\text{ideal bucket}
\]

1 $- \gamma$ fraction of bucket

\[
\begin{align*}
-\frac{n}{2B} & \quad 0 \quad \frac{n}{2B}
\end{align*}
\]
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Formally:

$$ \hat{G}_f := \frac{1}{Z} \left( \hat{H}_{f-U} + \hat{H}_{f+1-U} + \cdots + \hat{H}_{f+U} \right) $$

where $Z$ is a normalization factor.
To construct \( \hat{G} \):

1. sum up shifts \( \hat{H}_{-\Delta} \) over all \( \Delta \in [-U, U] \), where
   
   \[
   U = (1 - \gamma / 2) \frac{n}{2B}
   \]

2. normalize so that \( \hat{G}_0 = 1 \pm \delta \)

Formally:

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\hat{G}_f := \frac{1}{Z} \left( \hat{H}_{f-U} + \hat{H}_{f+1-U} + \ldots + \hat{H}_{f+U} \right)
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where \( Z \) is a normalization factor.

Upon inspection, \( Z = \sum_{f \in [n]} \hat{H}_f \) works.
Formally:

\[ \hat{G}_f := \frac{1}{Z} \left( \hat{H}_{f-U} + \hat{H}_{f+1-U} + \ldots + \hat{H}_{f+U} \right) \]

where \( Z \) is a normalization factor.

Upon inspection, \( Z = \sum_{f \in [n]} \hat{H}_f \) works.

(Flat region) For any \( f \in \left[ -(1 - \gamma) \frac{n}{2B}, (1 - \gamma) \frac{n}{2B} \right] \) (flat region) one has

\[ \hat{H}_{f-U} + \hat{H}_{f+1-U} + \ldots + \hat{H}_{f+U} \geq \sum_{f \in [-\gamma \frac{n}{4B}, \gamma \frac{n}{4B}]} \hat{H}_f \]

\[ \geq Z - \text{tail of } \hat{H} \]

\[ \geq Z - (\delta/n)n \geq Z - \delta \]
Formally:
\[
\hat{G}_f := \frac{1}{Z} \left( \hat{H}_{f-U} + \hat{H}_{f+1-U} + \ldots + \hat{H}_{f+U} \right)
\]
where \( Z \) is a normalization factor.

Upon inspection, \( Z = \sum_{f \in [n]} \hat{H}_f \) works.

Indeed, for any \( f \not\in \left[ -\frac{n}{2B}, \frac{n}{2B} \right] \) (zero region) one has
\[
\hat{H}_{f-U} + \hat{H}_{f+1-U} + \ldots + \hat{H}_{f+U} \leq \sum_{f > \gamma \frac{n}{4B}} \hat{H}_f
\]
\[
\leq \text{tail of } \hat{H} \leq (\delta/n)n \leq \delta
\]
Flat window function

1 − γ fraction of bucket

ideal bucket

How large is support of $\hat{G} := \frac{1}{Z} \left( \hat{H}_{-U} + \ldots + \hat{H}_{+U} \right)$?
Flat window function

1 − γ fraction of bucket

ideal bucket

How large is support of \( \hat{G} := \frac{1}{Z} \left( \hat{H}_{-U} + \ldots + \hat{H}_{+U} \right) \)?

By time shift theorem for every \( q \in [n] \)

\[
G_q := H_q \cdot \frac{1}{Z} \sum_{j=-U}^{U} \omega^{qj}
\]
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By time shift theorem for every $q \in [n]$:

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Support of $G$ a subset of support of $H$!
Flat window functions – construction