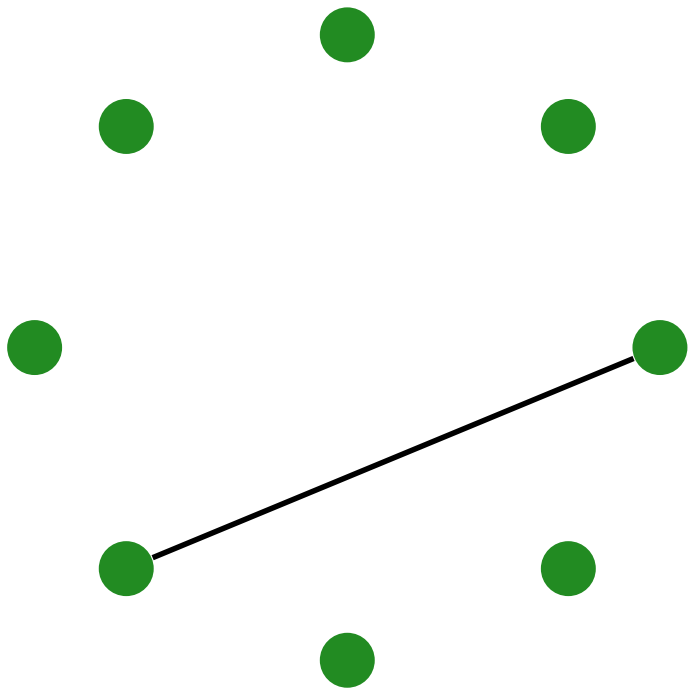
Graph Sketching

Michael Kapralov
EPFL

Streaming model

- edges of G arrive in an arbitrary order in a stream
- algorithm can only use $\tilde{O}(n)$ space
- several passes over the stream



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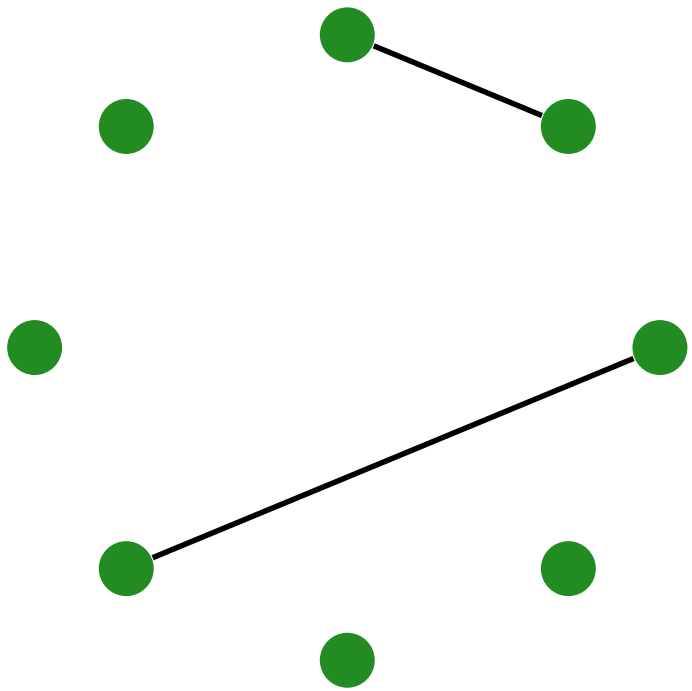

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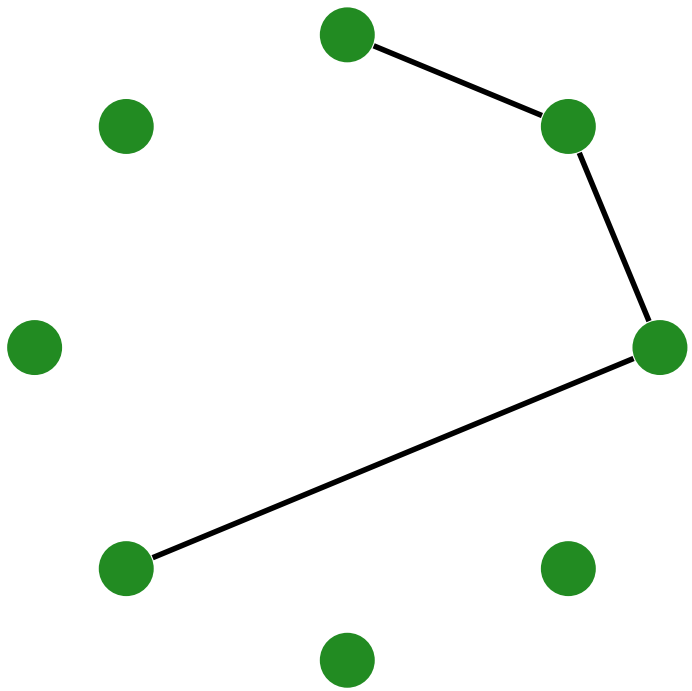

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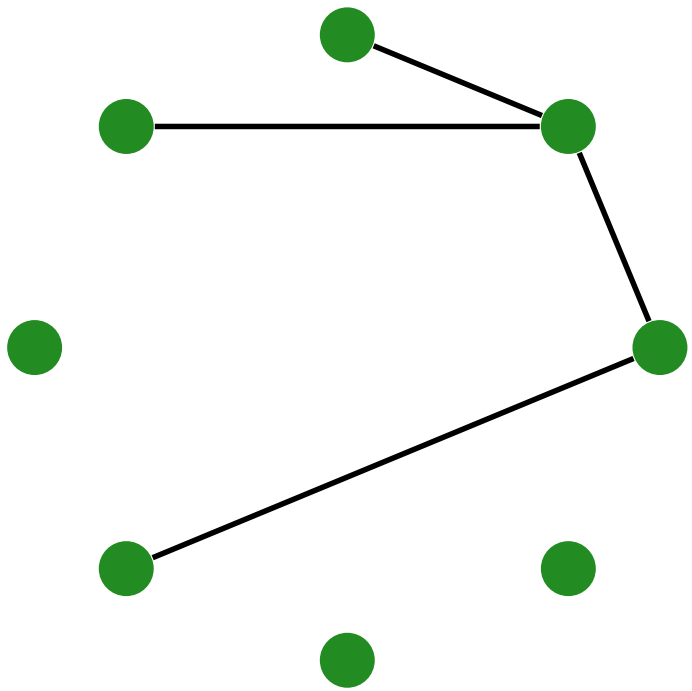

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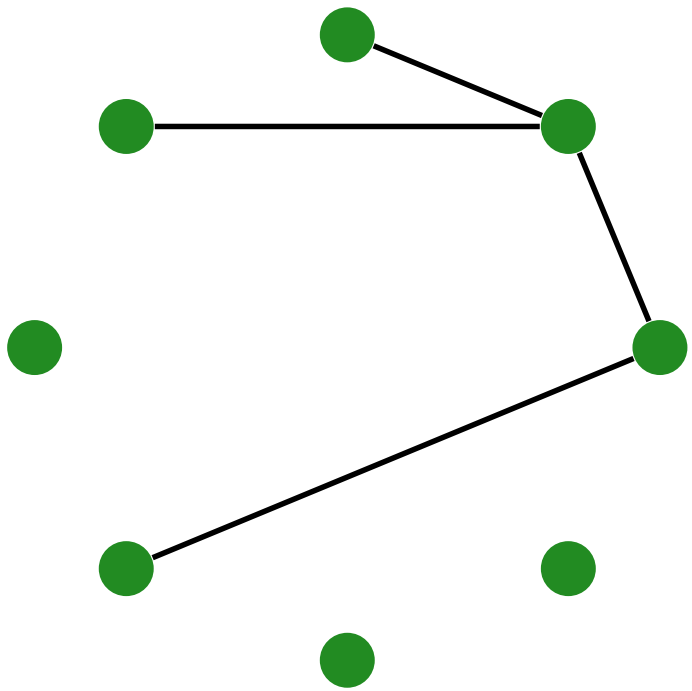

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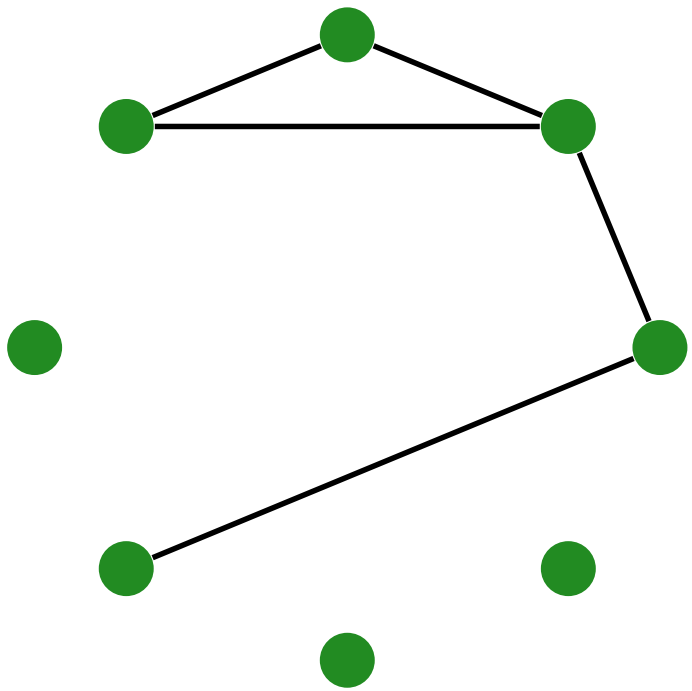

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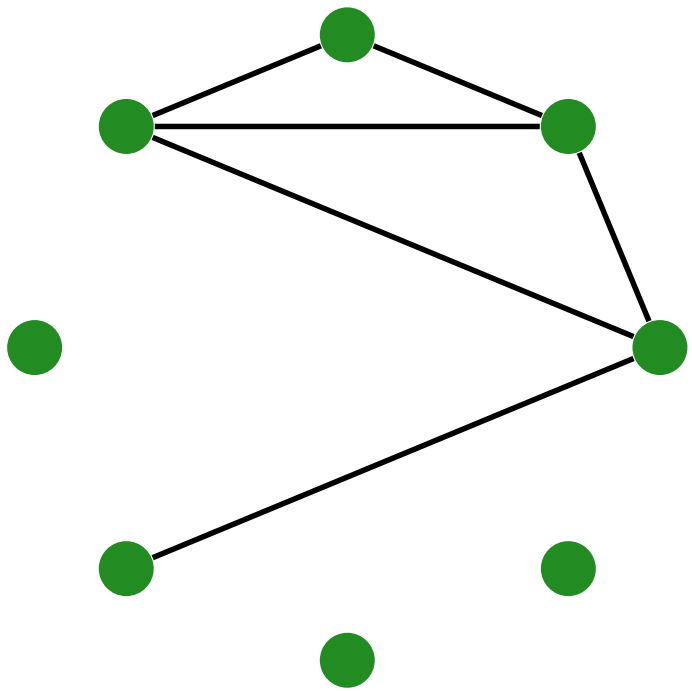

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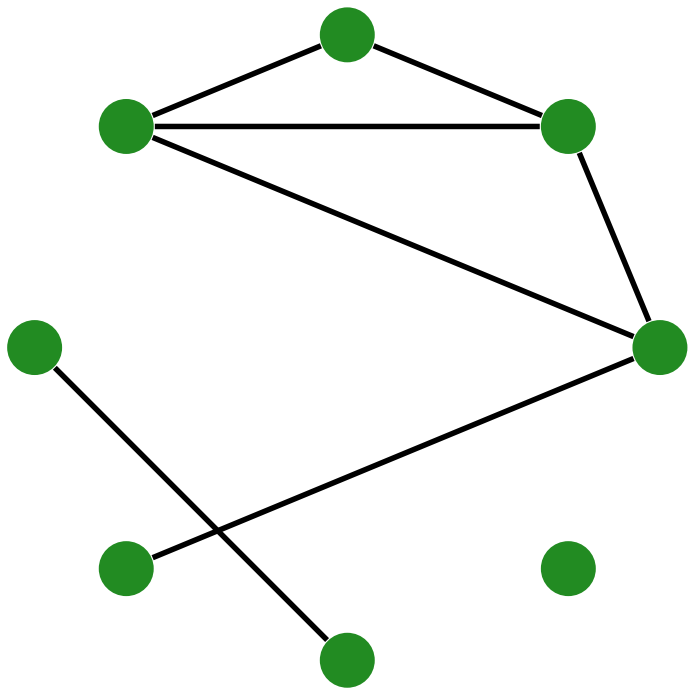

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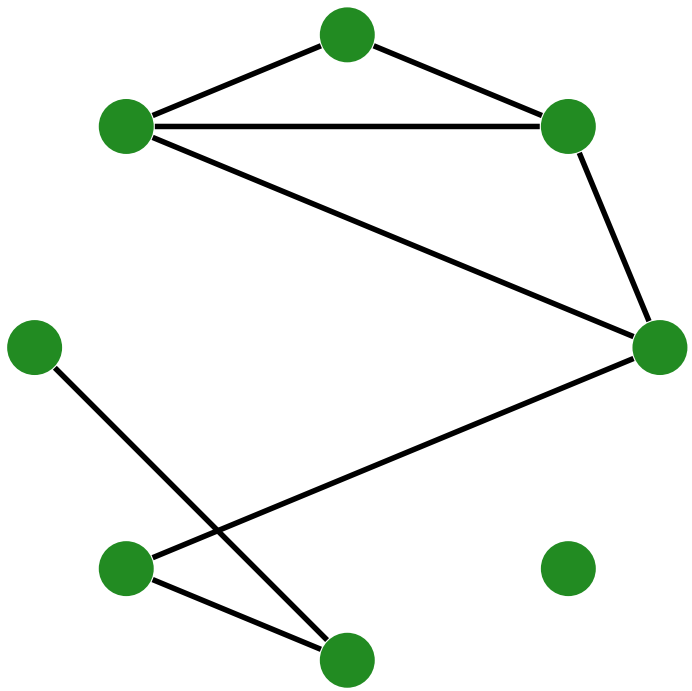

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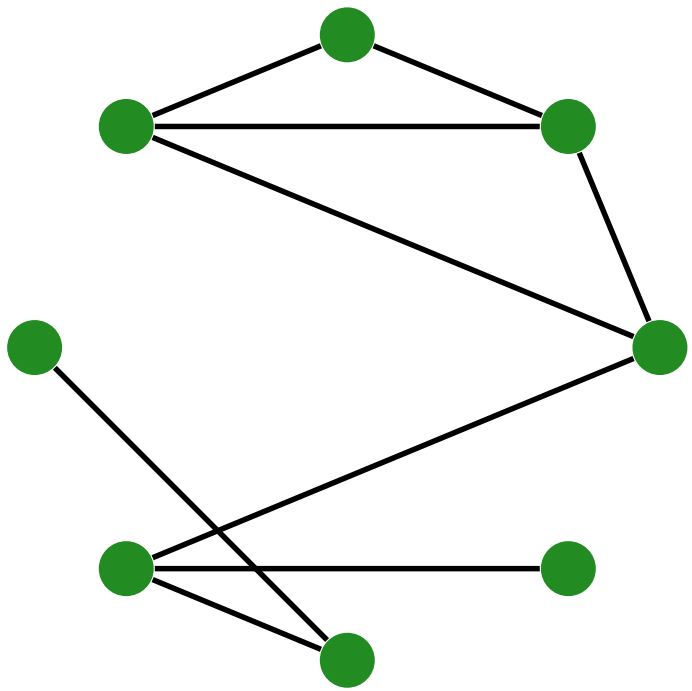

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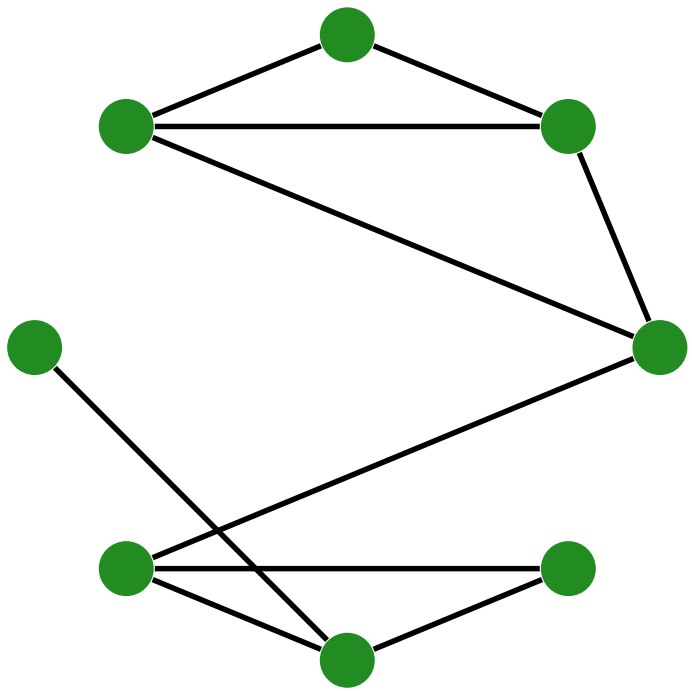

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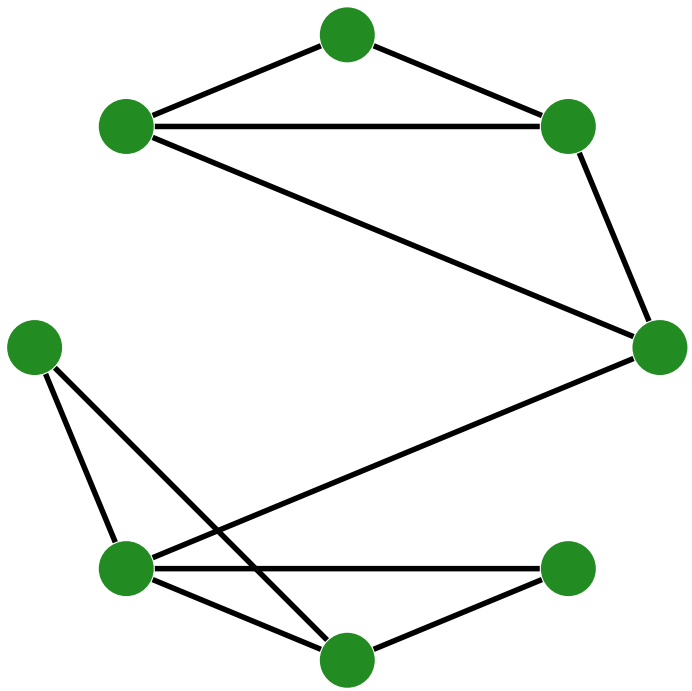

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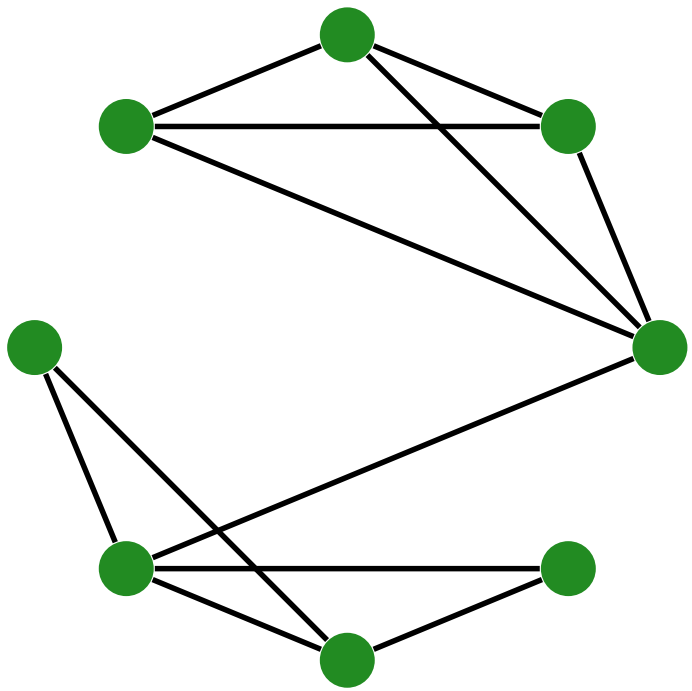

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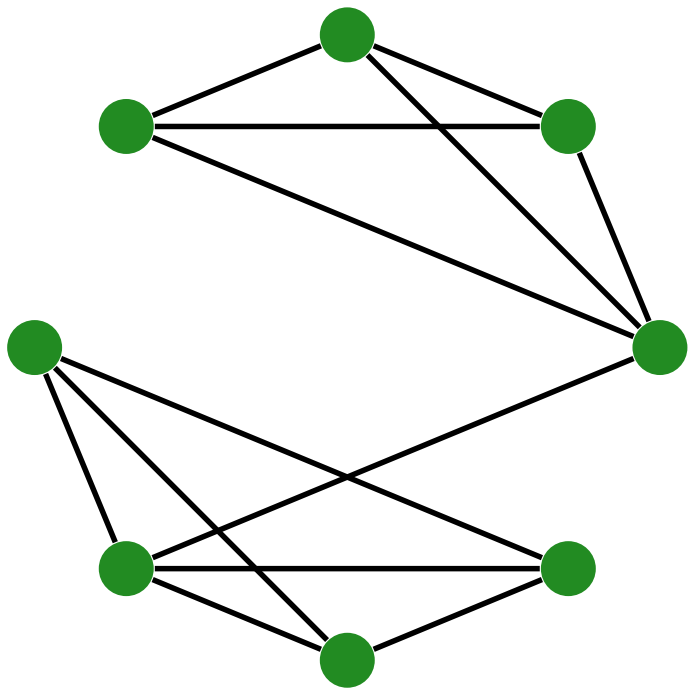

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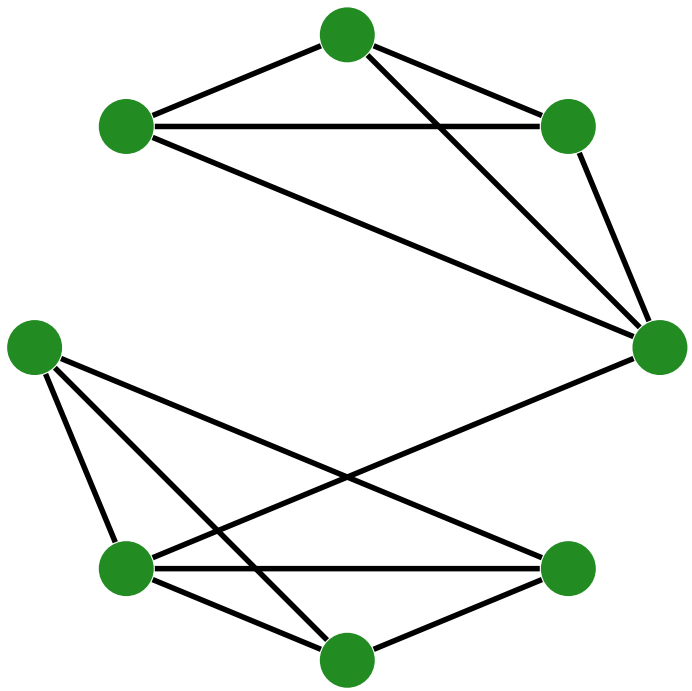

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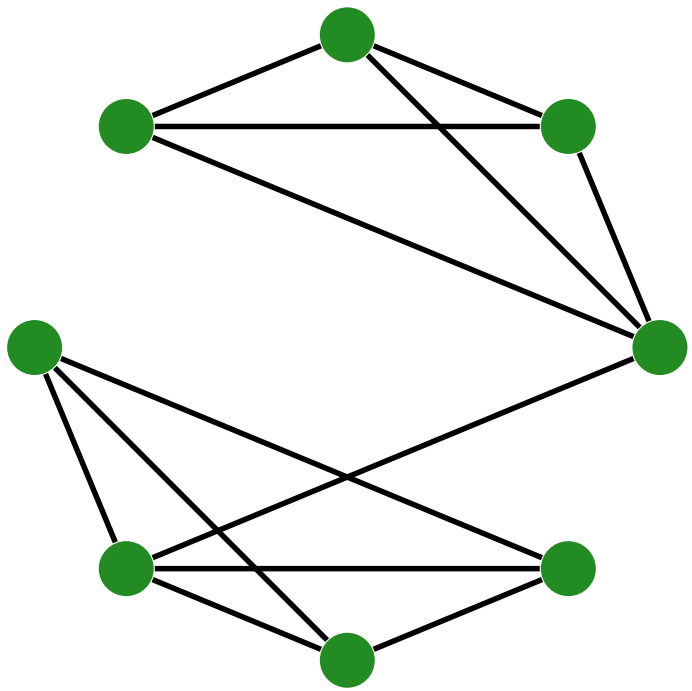

Insertion-only stream

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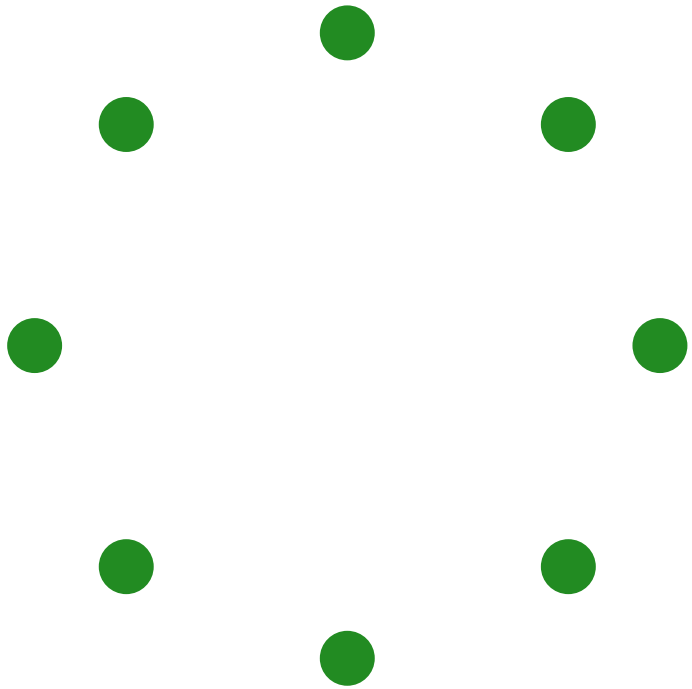
Spanning trees in insertion-only streams



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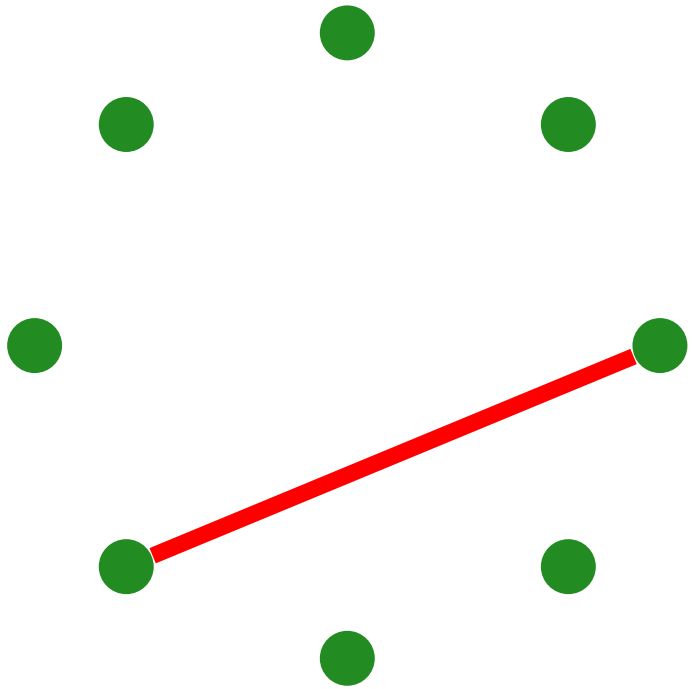

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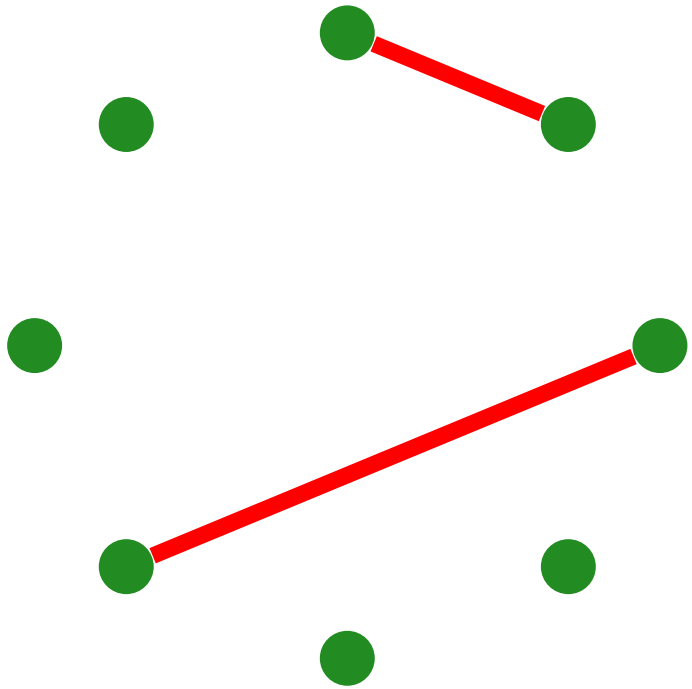

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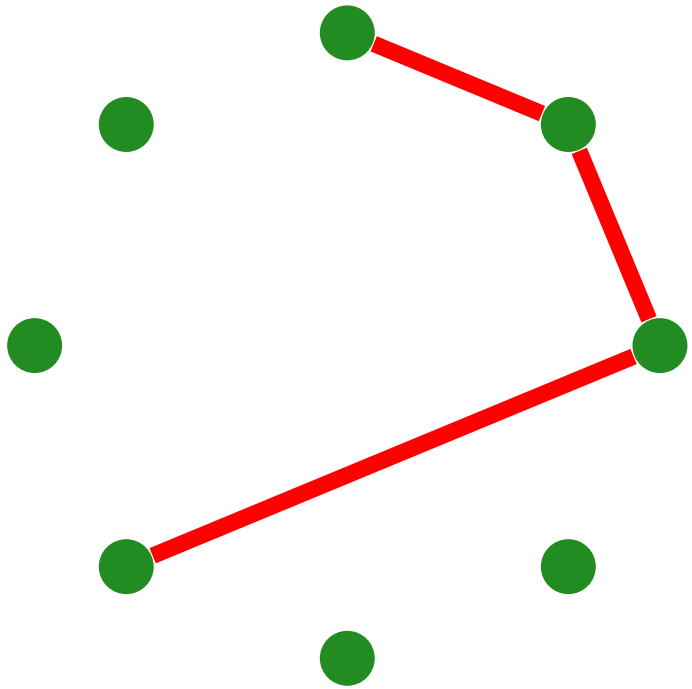

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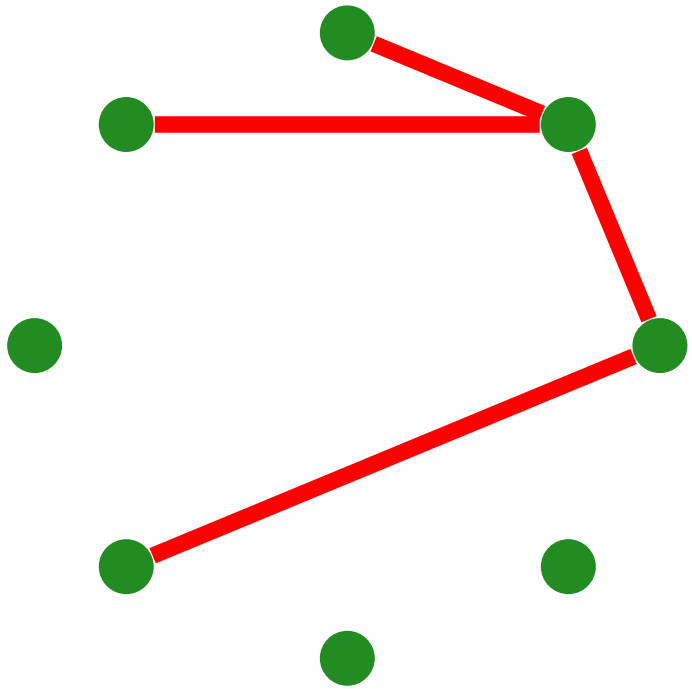

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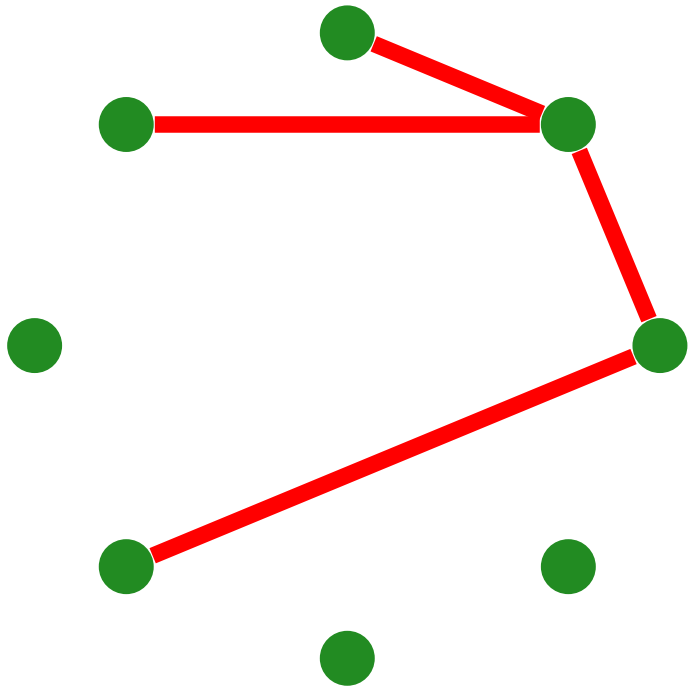

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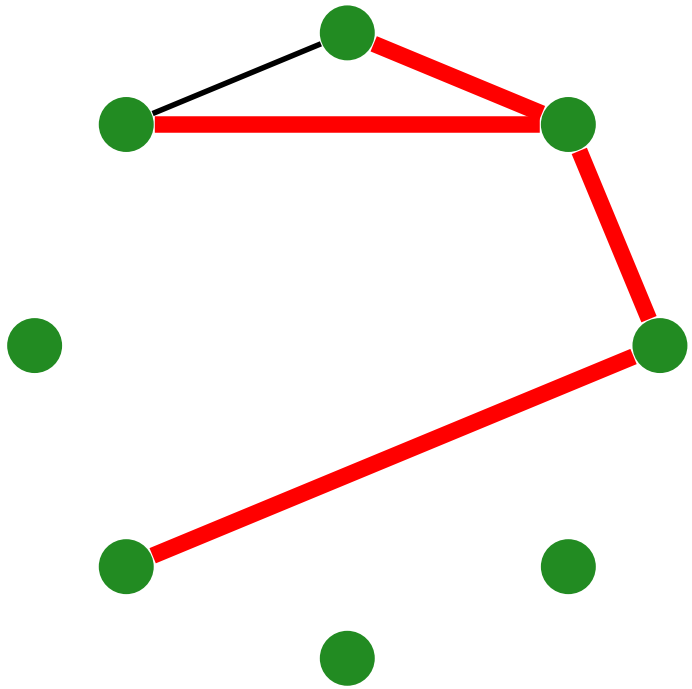

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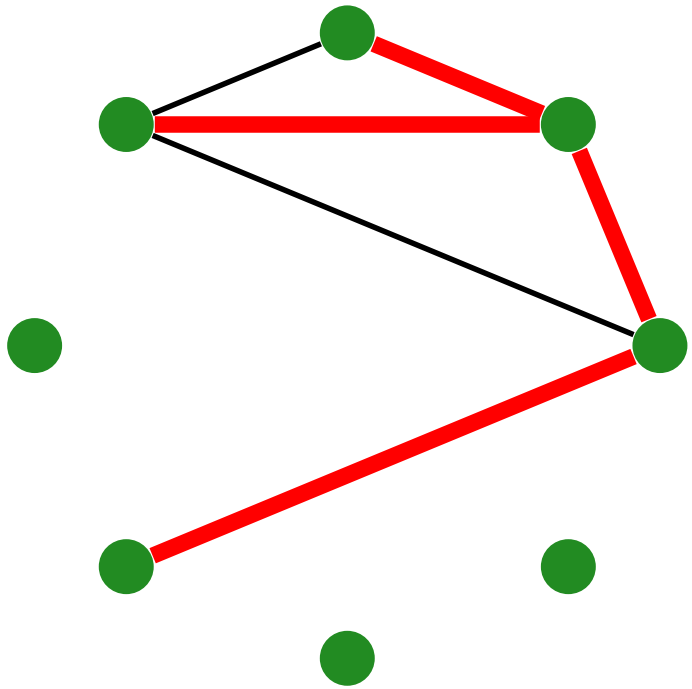

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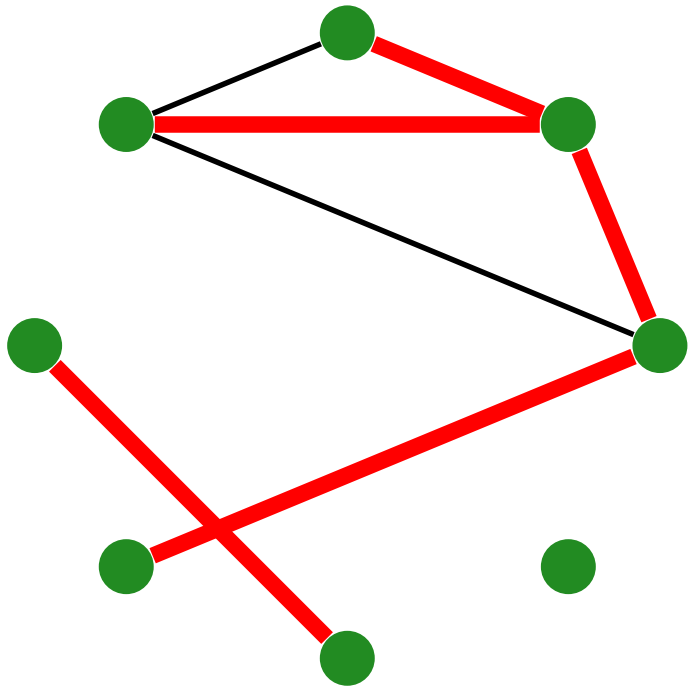

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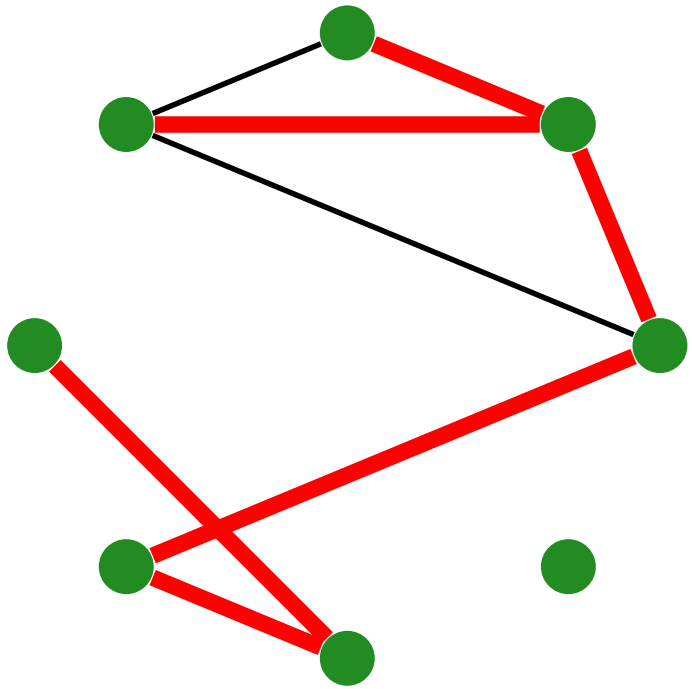

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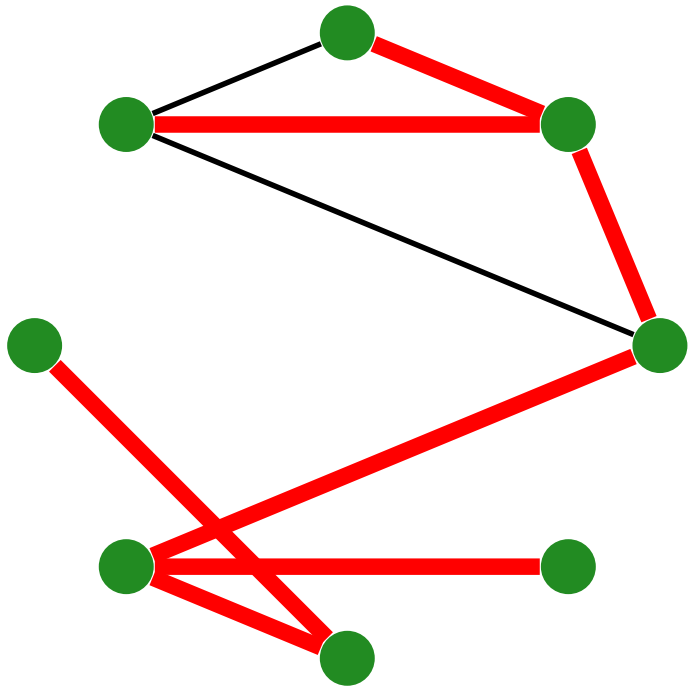

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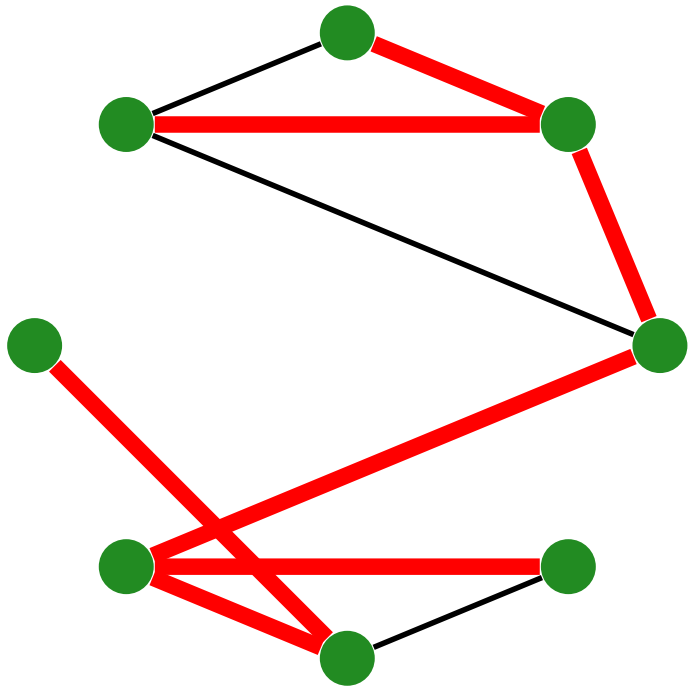

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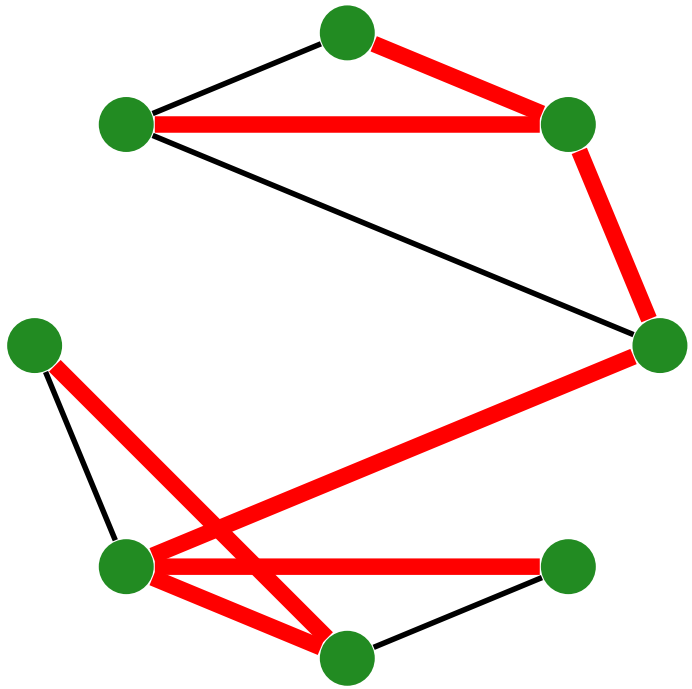

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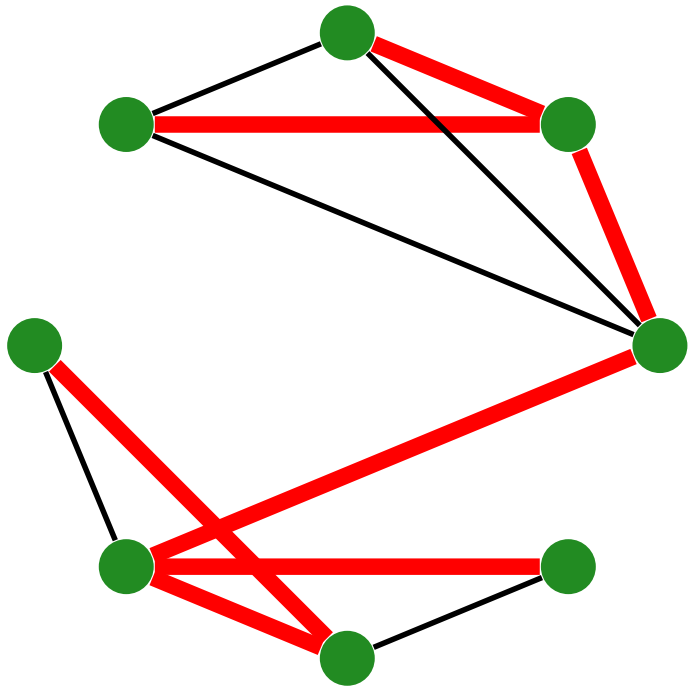

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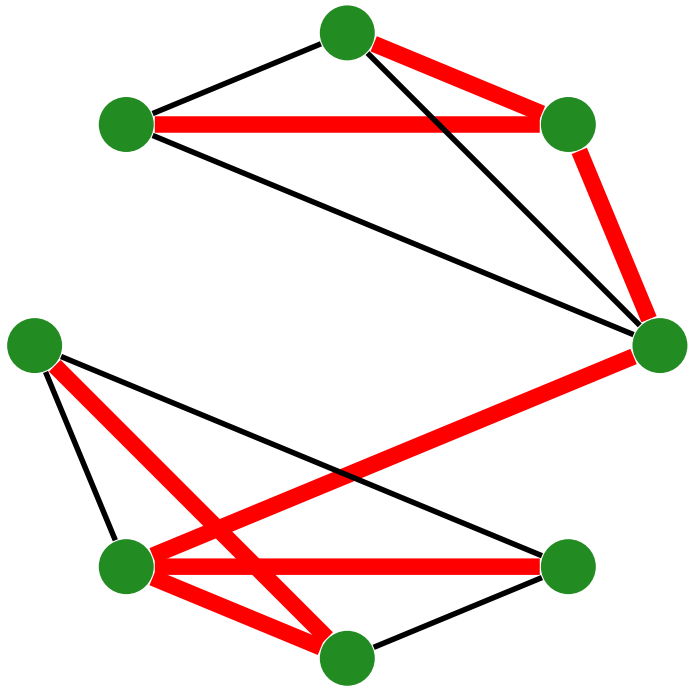

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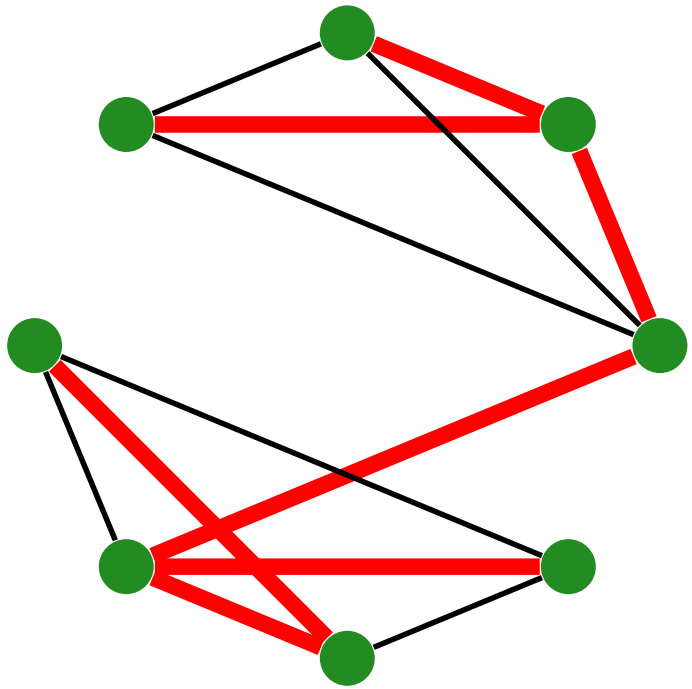

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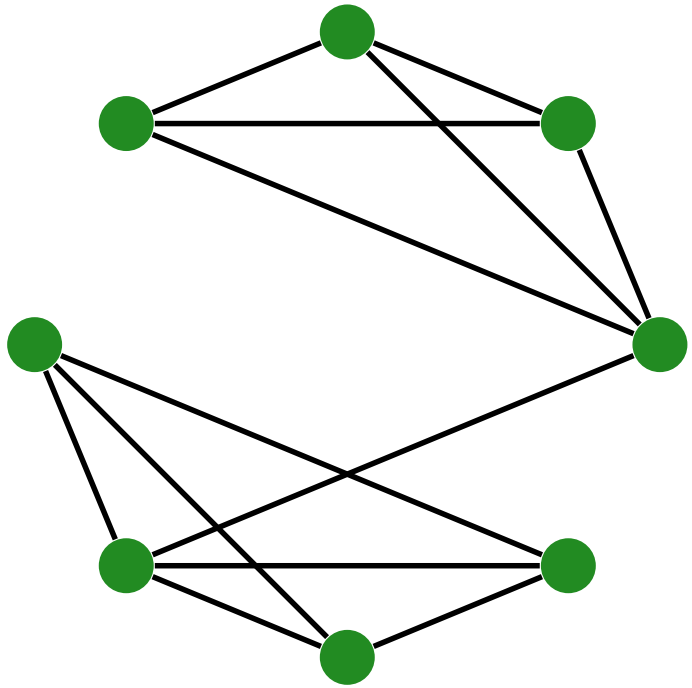
Many modern networks evolve over time, edges both inserted and deleted

Construct spanning trees in dynamic streams in small space?

What if we have deletions?



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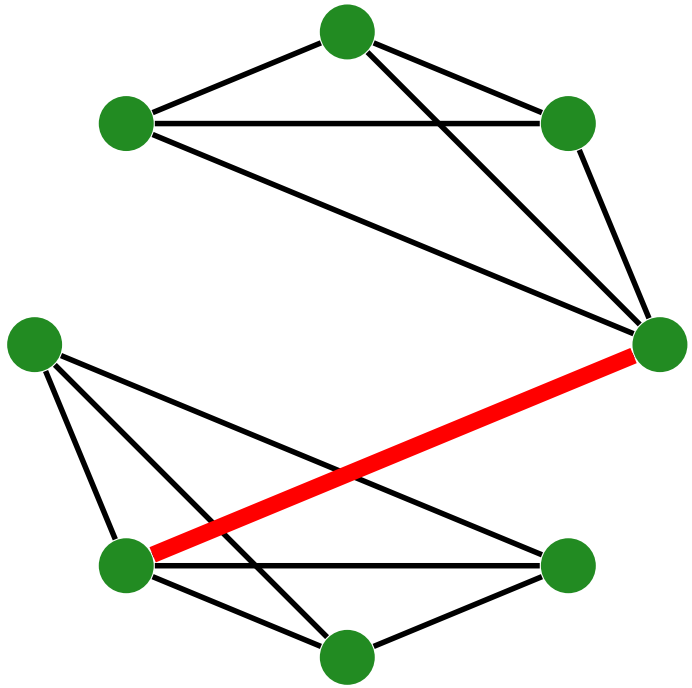

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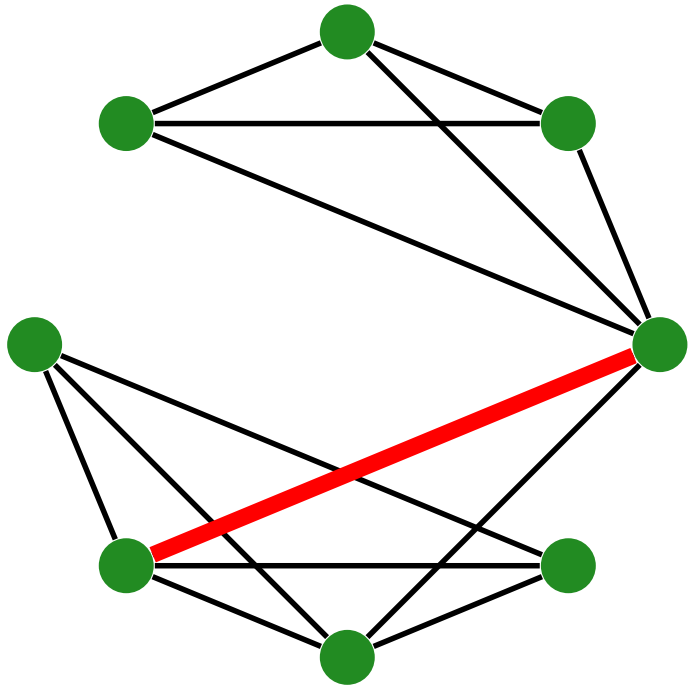

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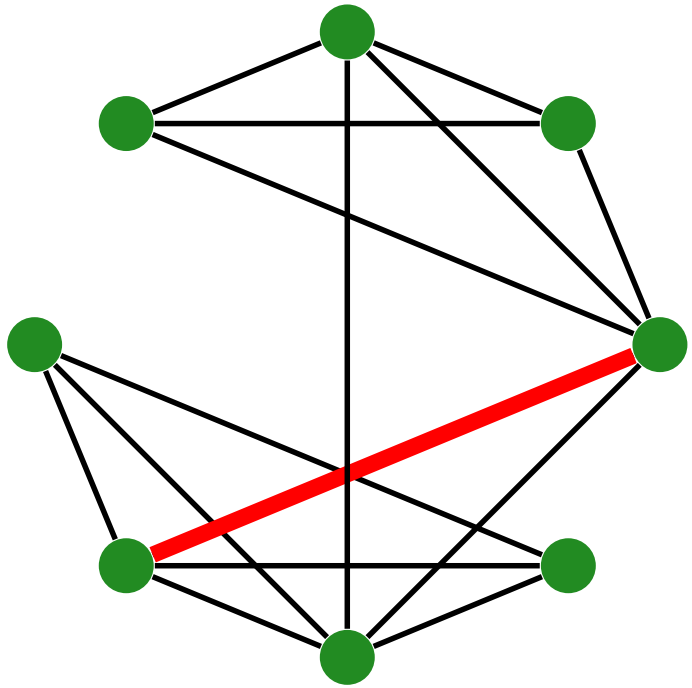

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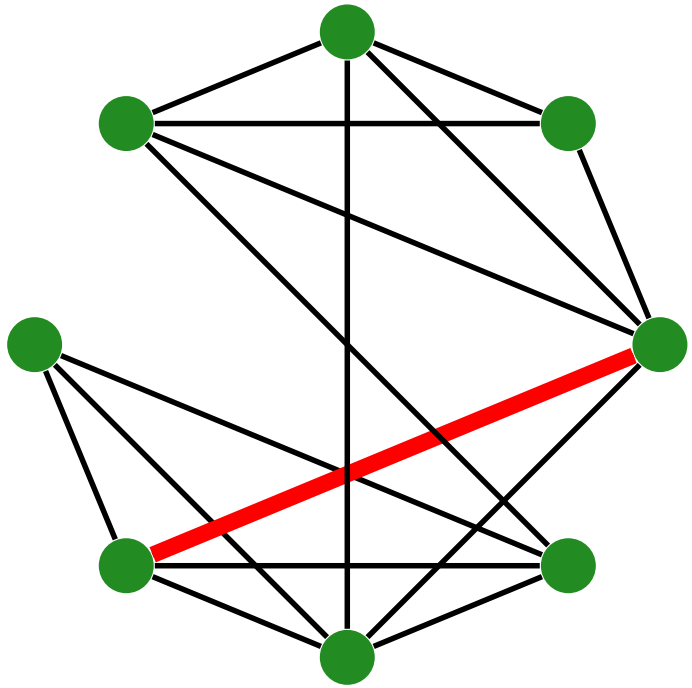

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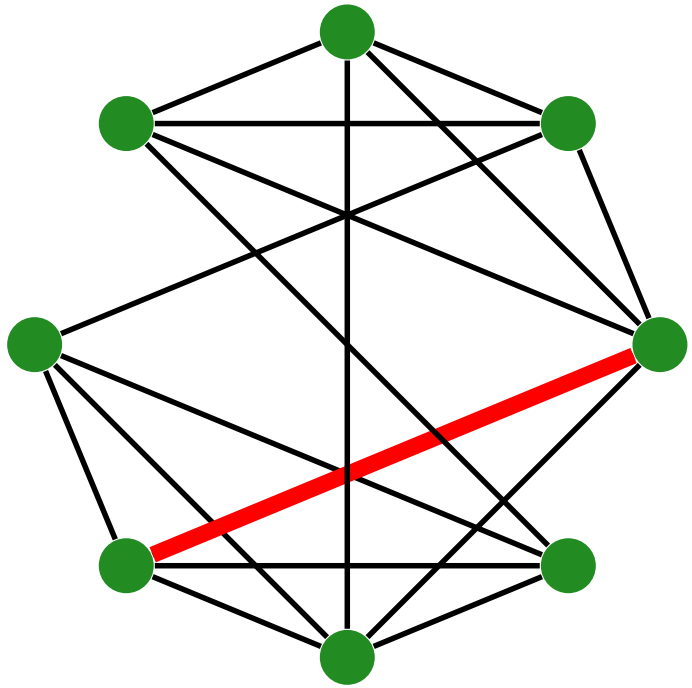

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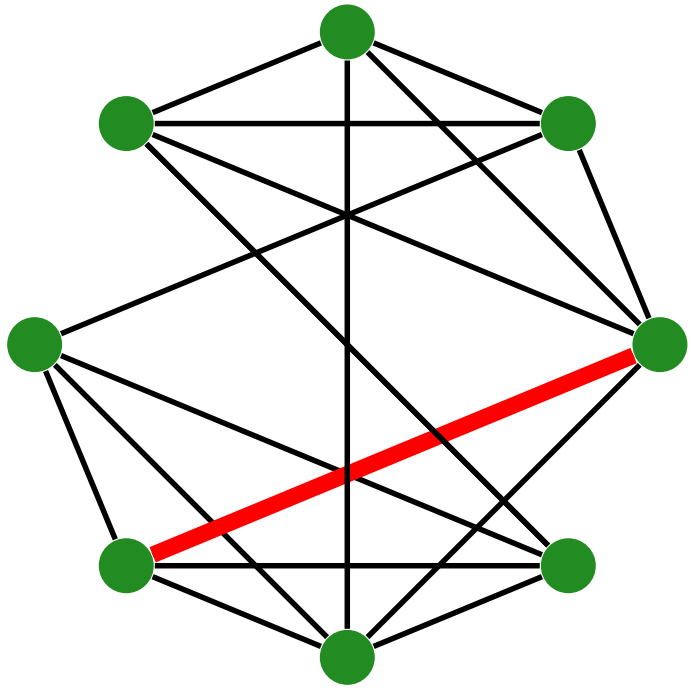

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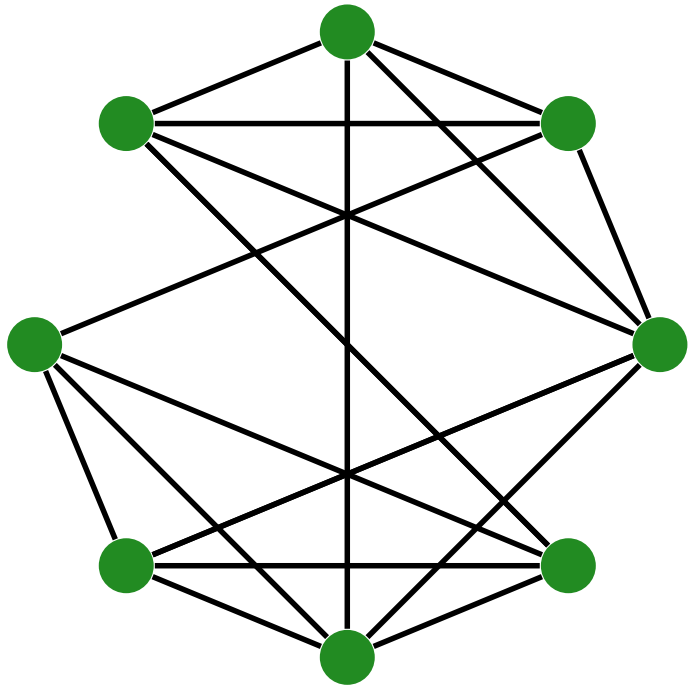

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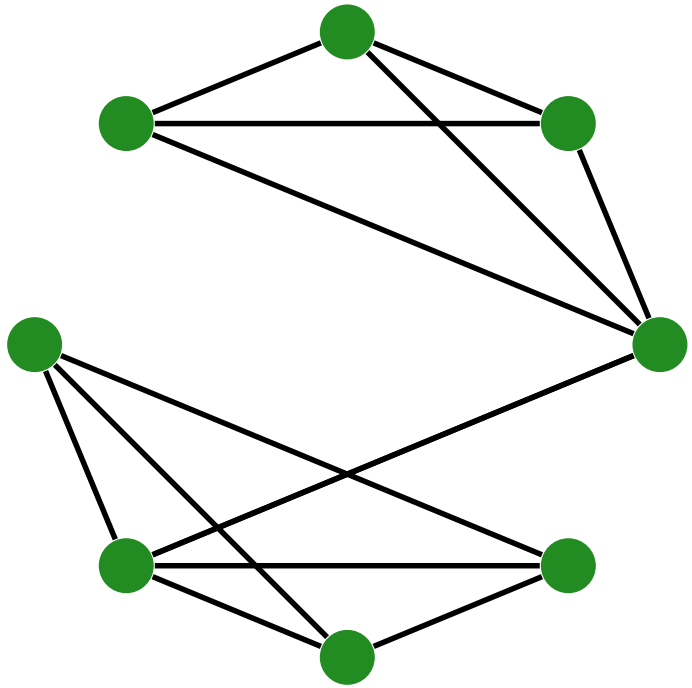

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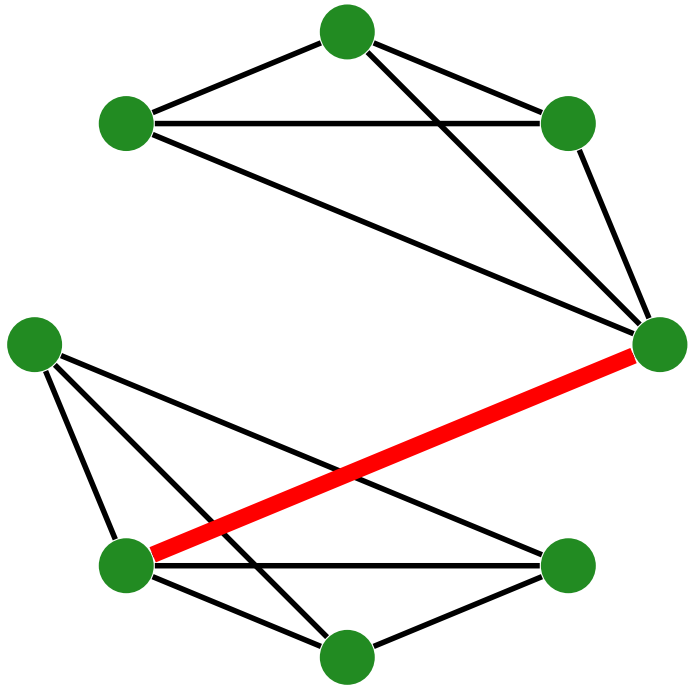
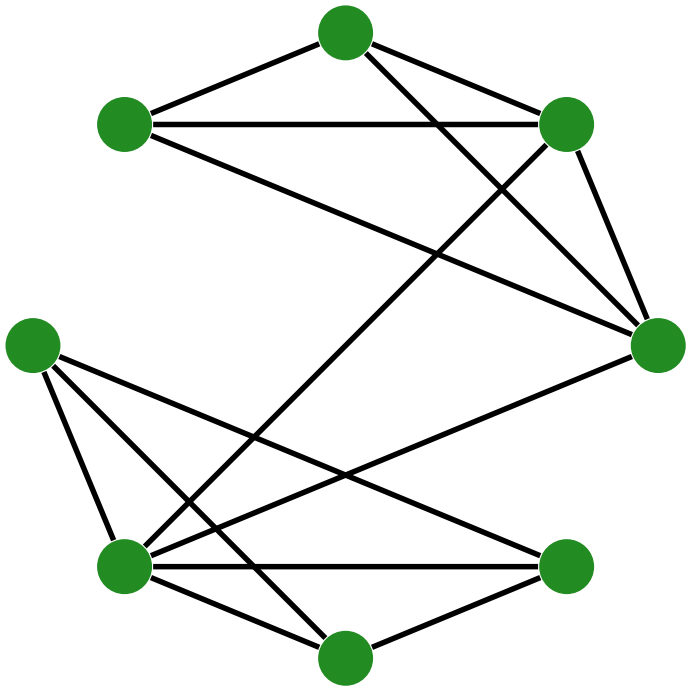


Very different algorithms are needed...

Main idea: sketch the edge-vertex incidence matrix of G!



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$\binom{n}{2} \times n$

Each row of B=potential edge in G.

If $e = (u, v) \in E$, then $b_e = \chi_u - \chi_v$, otherwise $b_e = 0$

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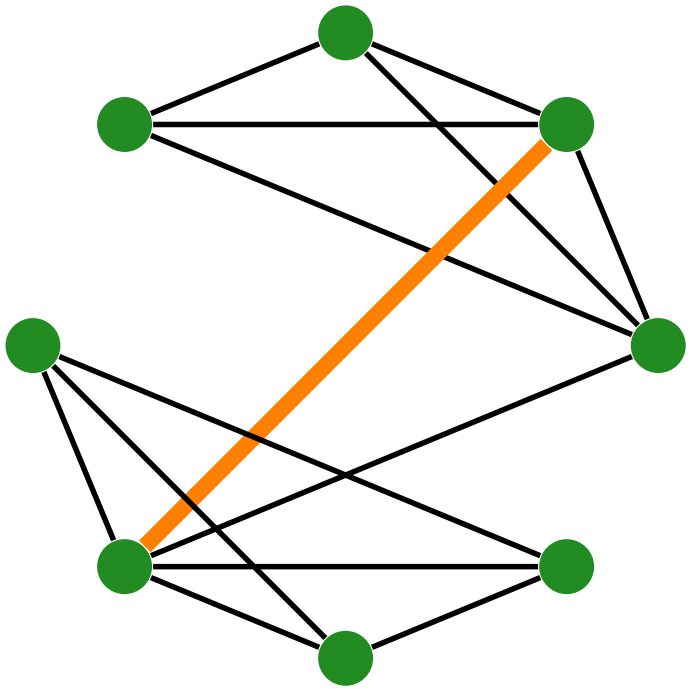

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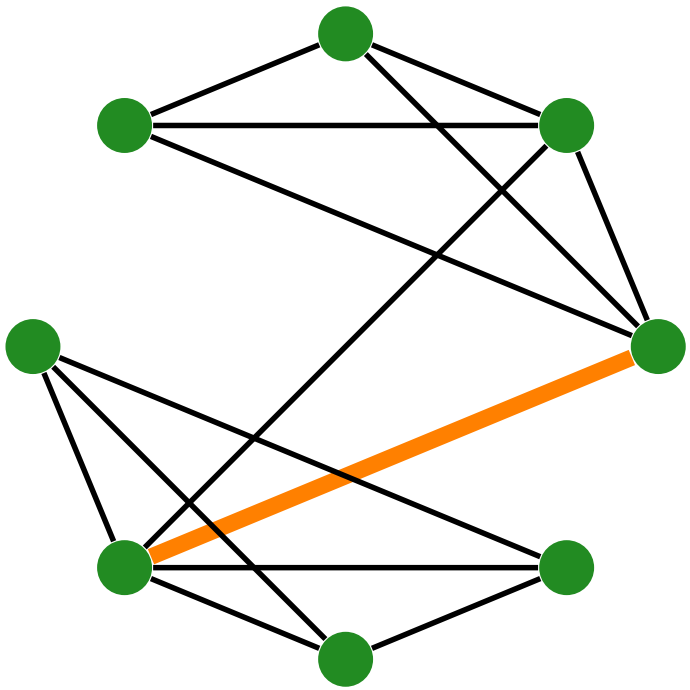

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$$\delta(C) := E \cap (C \times (V \setminus C))$$

denote the set of edges crossing the cut and let

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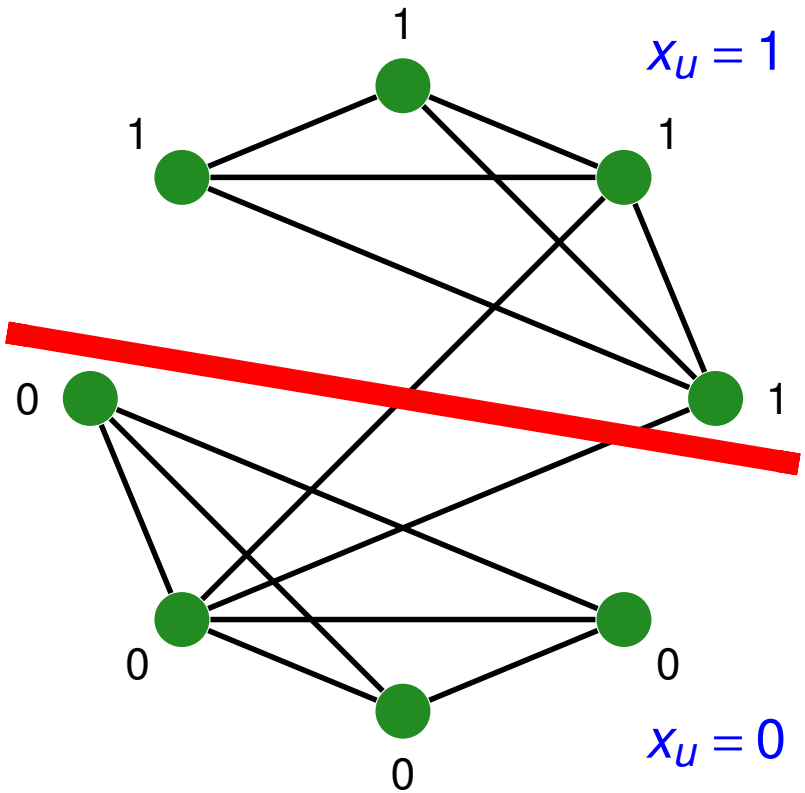

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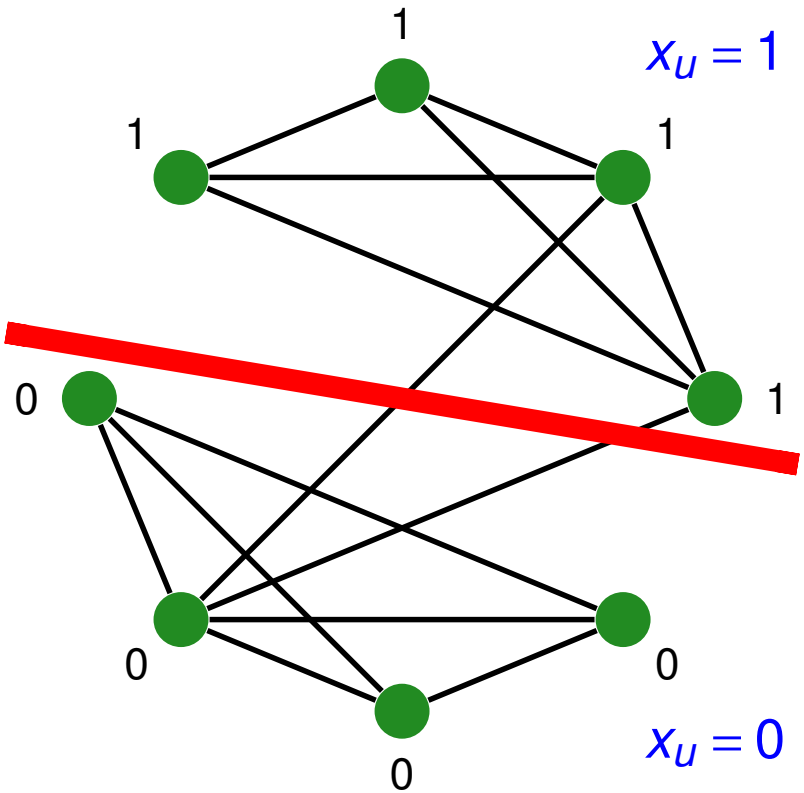
Proof by picture.



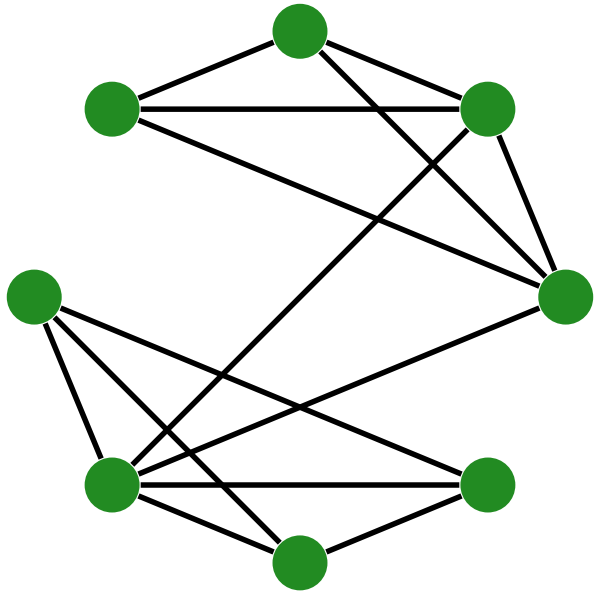
$$B = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \end{bmatrix}$$

$\binom{n}{2} \times n$

- Graph streaming
- Connectivity via sketching

- Graph streaming
- **Connectivity via sketching**

A simple algorithm for connectivity



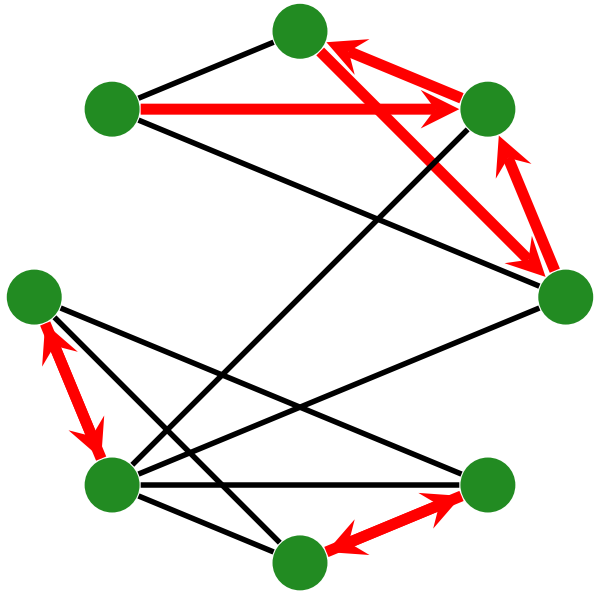
$F_0 \leftarrow \emptyset$
$\mathscr{C}_0 \leftarrow \bigcup_{u \in V} \{u\}$ (current components)
for $t = 0$ **to** T

  for each $C \in \mathscr{C}_t$
    choose an edge in $\delta(C)$ (nonzero of $B \cdot \mathbf{1}_C$)

  $F_{t+1} \leftarrow F_t \cup \{\text{spanning forest on selected edges}\}$
  $\mathscr{C}_{t+1} \leftarrow \{\text{new connected components}\}$

return F_{T+1}

A simple algorithm for connectivity



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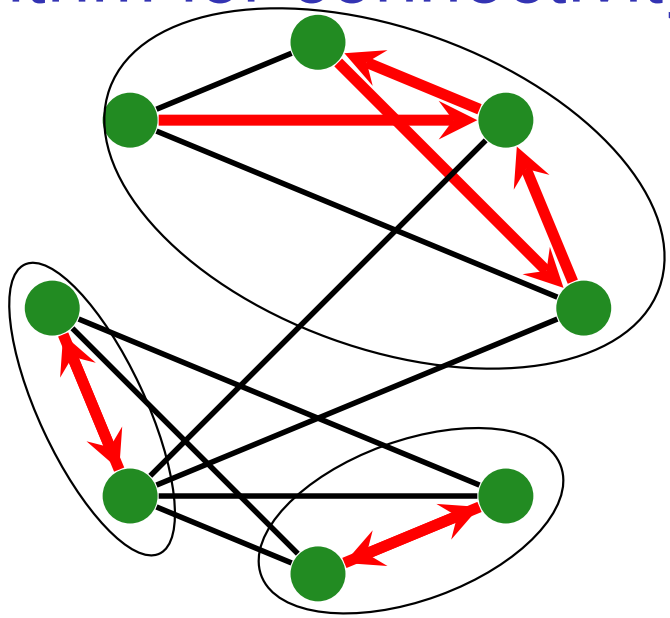

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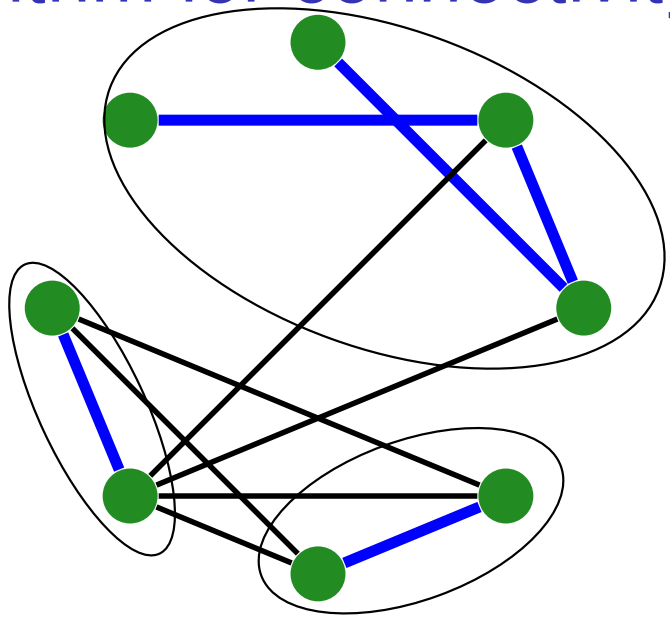

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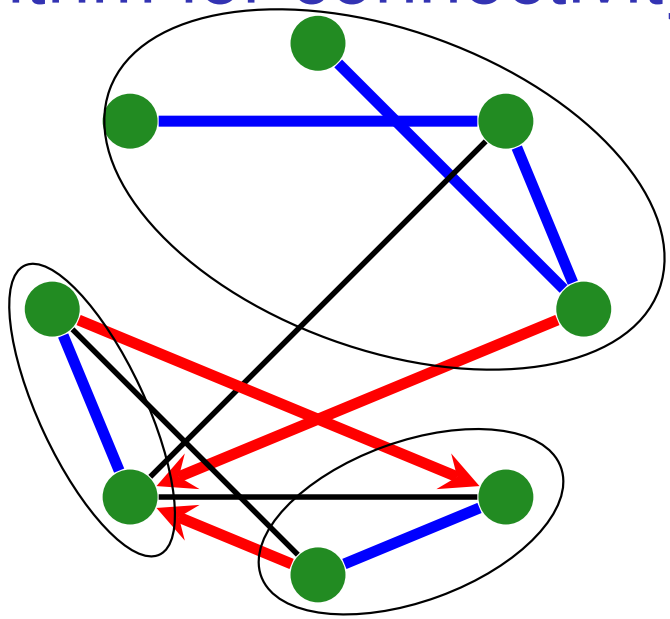

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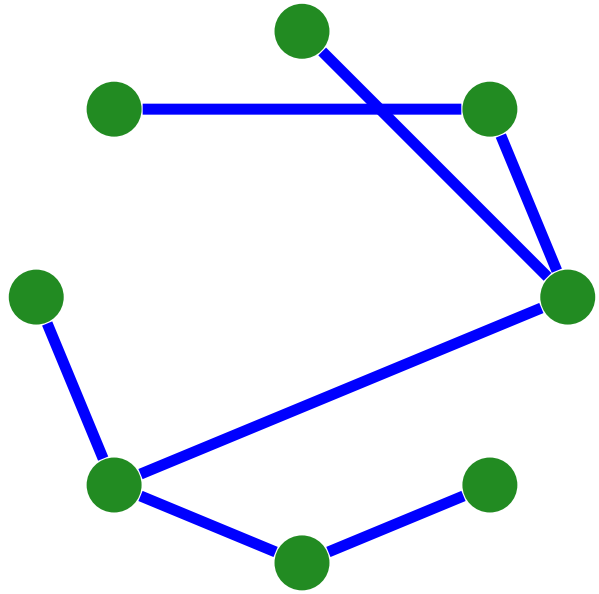

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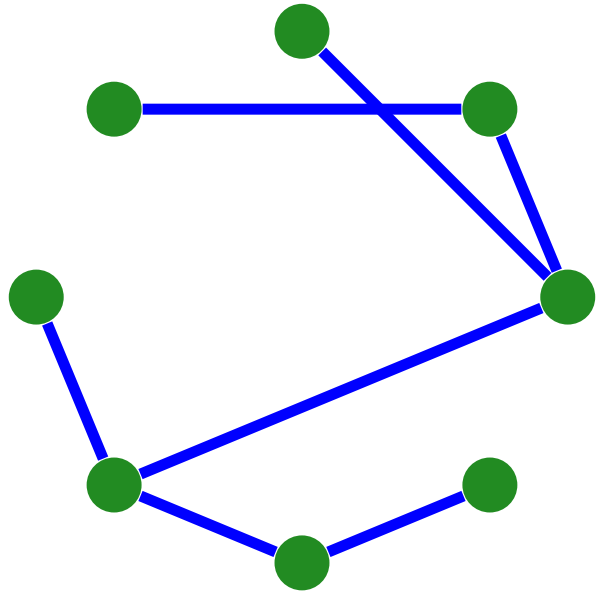

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(?? nonzero of $B \cdot \mathbf{1}_C$??)

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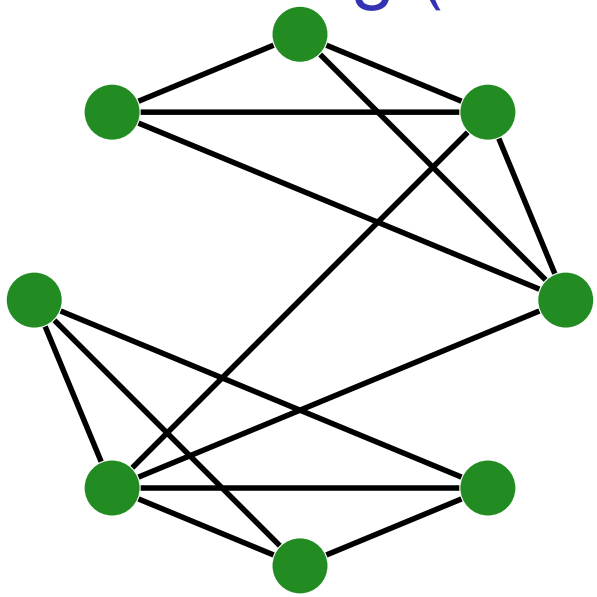
Connectivity via sketching (Ahn-Guha-McGregor'12)



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$S^0, \ldots, S^T \leftarrow \ell_0$-samplers
Maintain $S^0 B, \ldots, S^T B$

Run $Dec(S^t B \cdot \mathbf{1}_C)$

Some remarks

The result of Ahn, Guha, McGregor'12 independently and simultaneously obtained by Kapron, King'12

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Tightening parameters, get a sketch with bit complexity $O(n \log^3 n)$

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A: adaptivity

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Tightening parameters, get a sketch with bit complexity $O(n \log^3 n)$

Q: why did we need several sketches $S^0, \ldots, S^T$?

A: adaptivity

Very surprising: decoding is adaptive ($T = O(\log n)$ rounds), but sketch is not

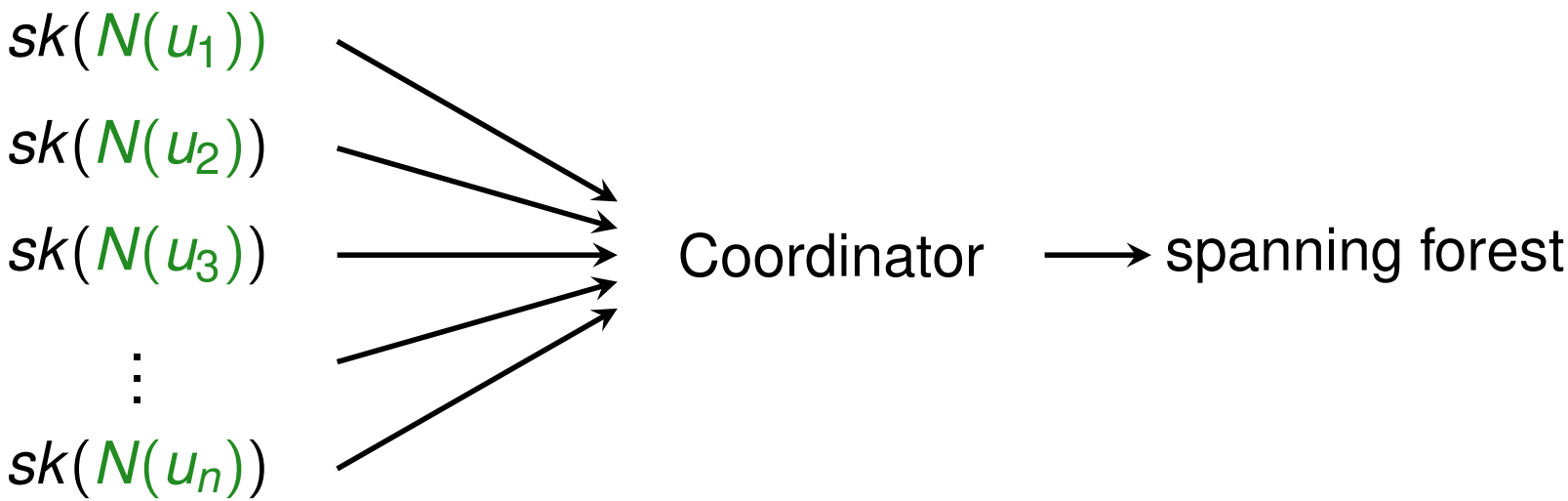
Nelson, Yu'19, Yu'21 – $\Omega(n \log^3 n)$ bits of space are needed

Input: Each $u \in V$ holds list of its neighbors $N(u)$

Each $u \in V$ sends a sketch of $N(u)$ to coordinator

(assume shared randomness)

Output: Spanning forest of G



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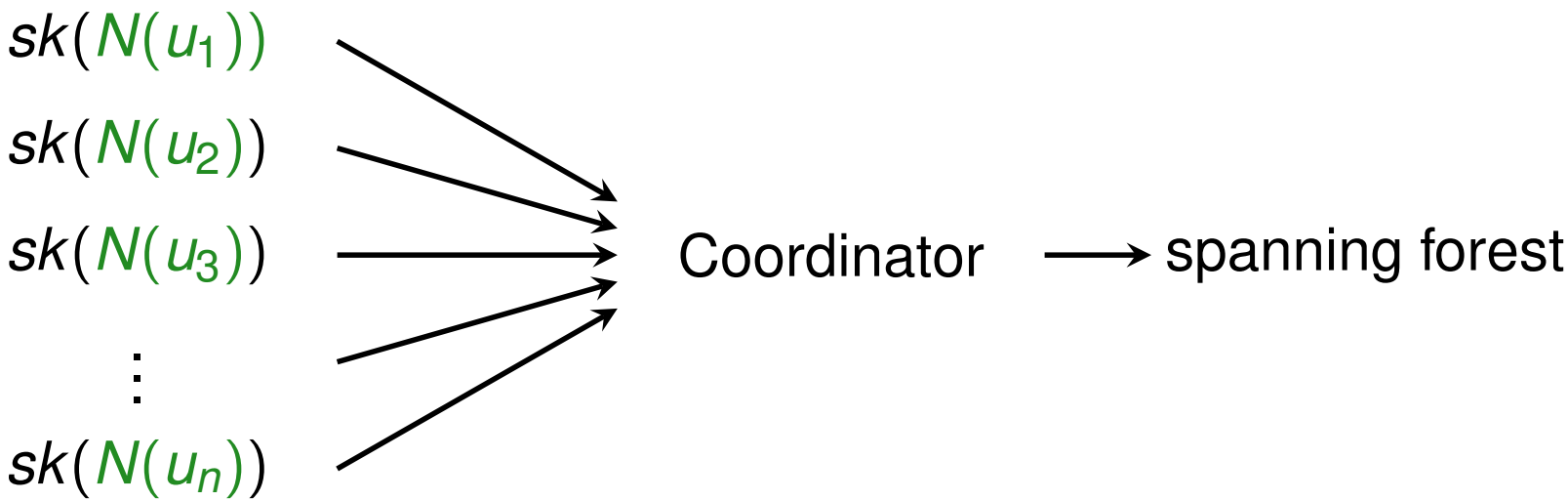

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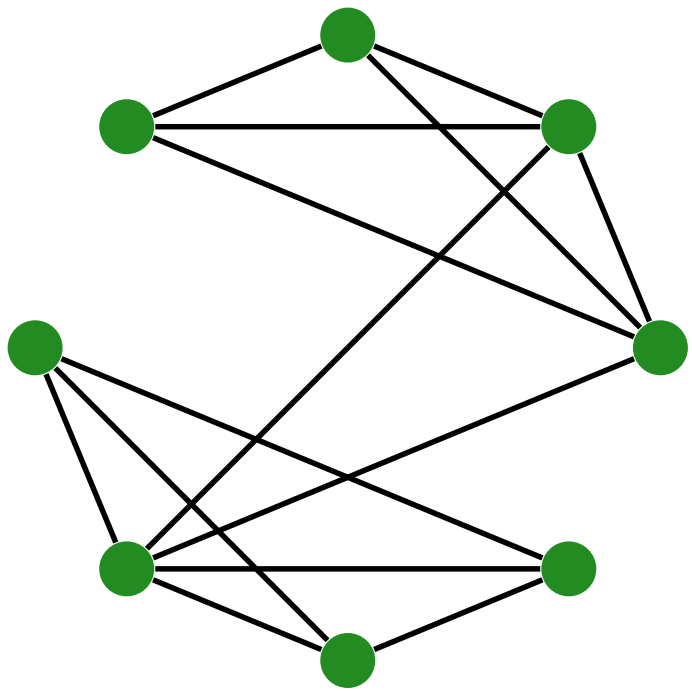
Nelson, Yu'19, Yu'21 – $\Omega(n \log^3 n)$ bits of communication are needed

Spectral sparsification of graphs

Sparsification

- $G = (V, E)$ an undirected graph, $|V| = n, |E| = m$
- Find sparse subgraph G' of G that approximates G



G

Sparsification

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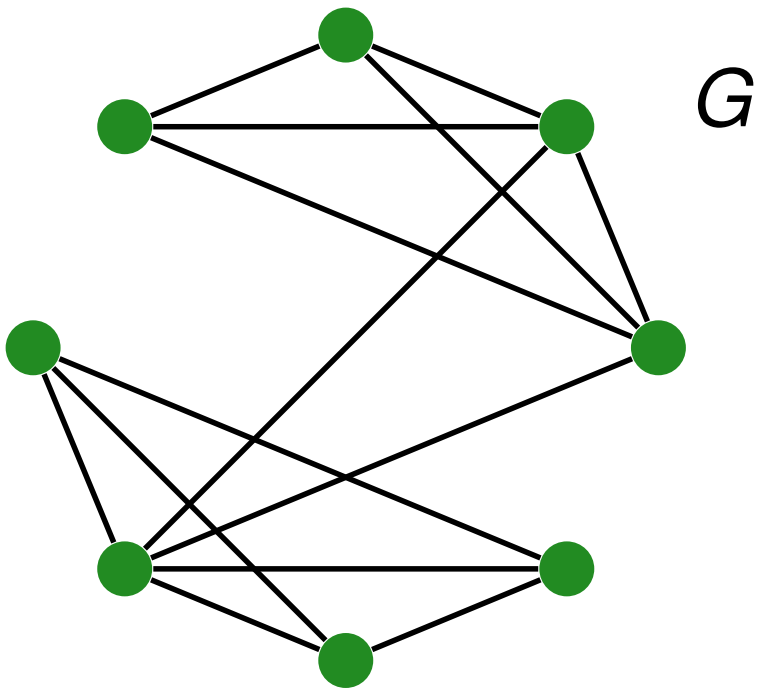
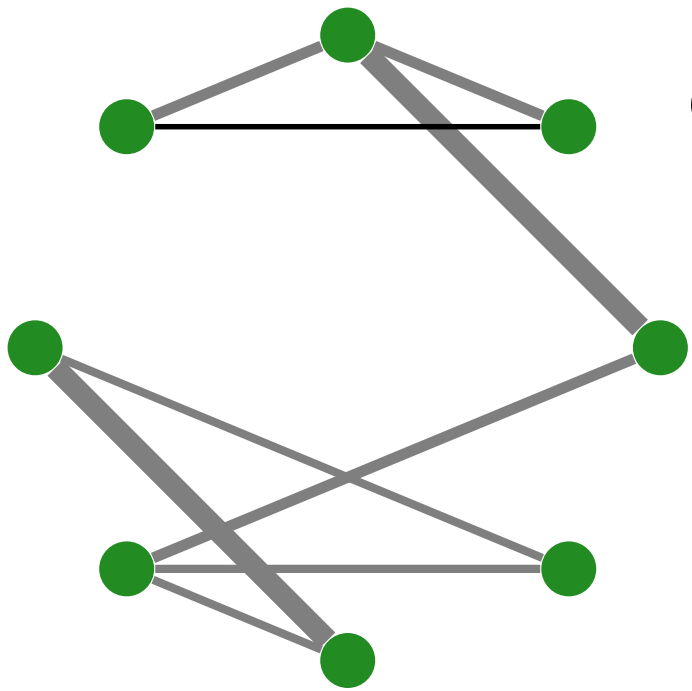


G



G'

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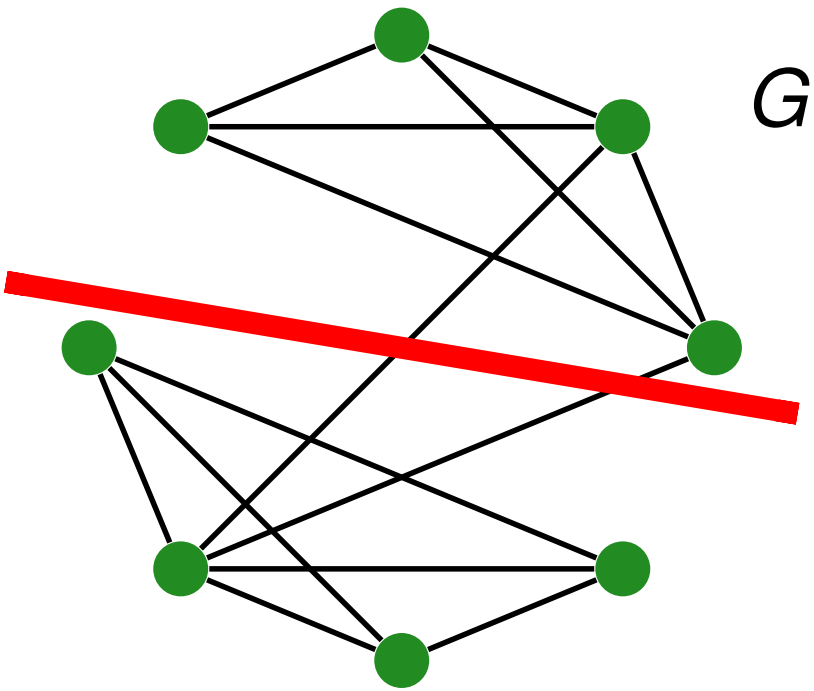

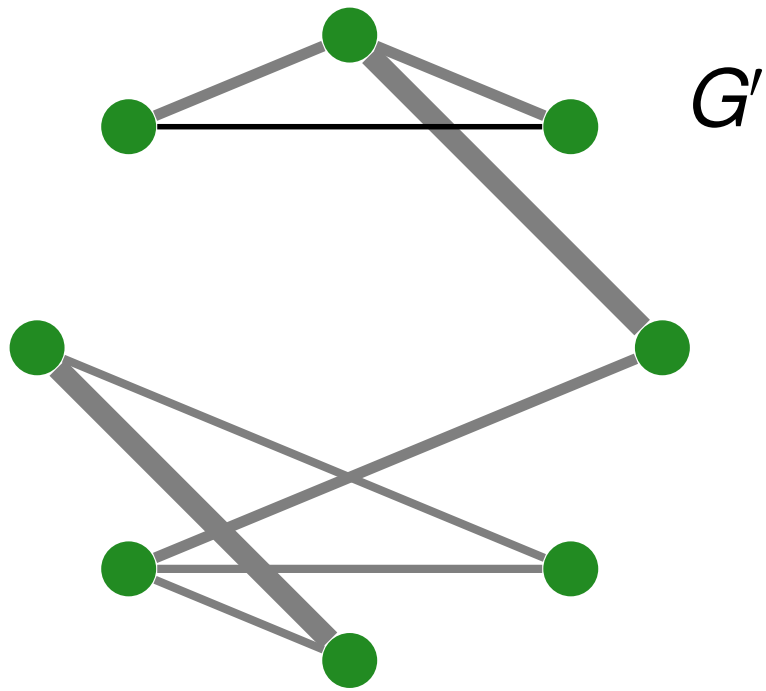

Sparsification

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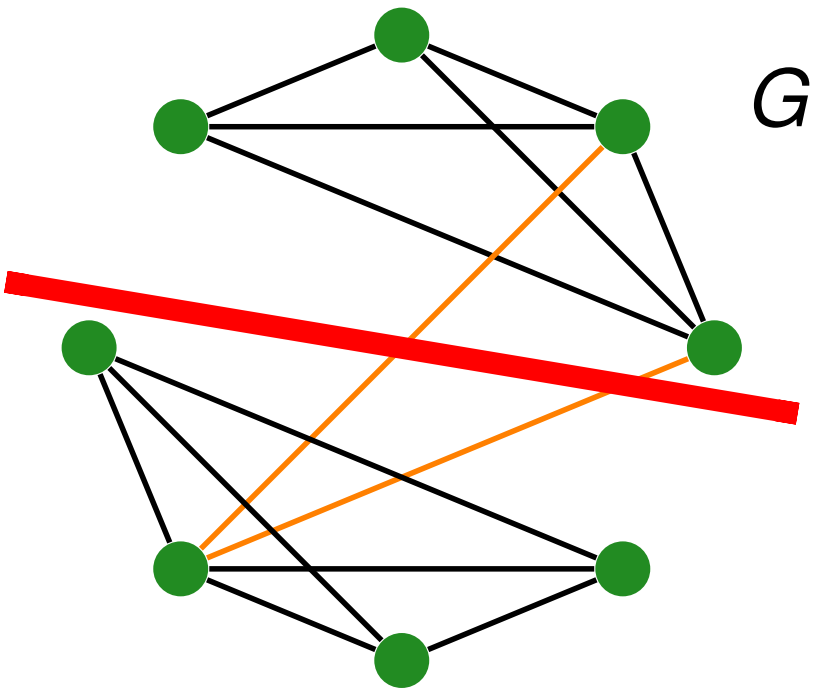

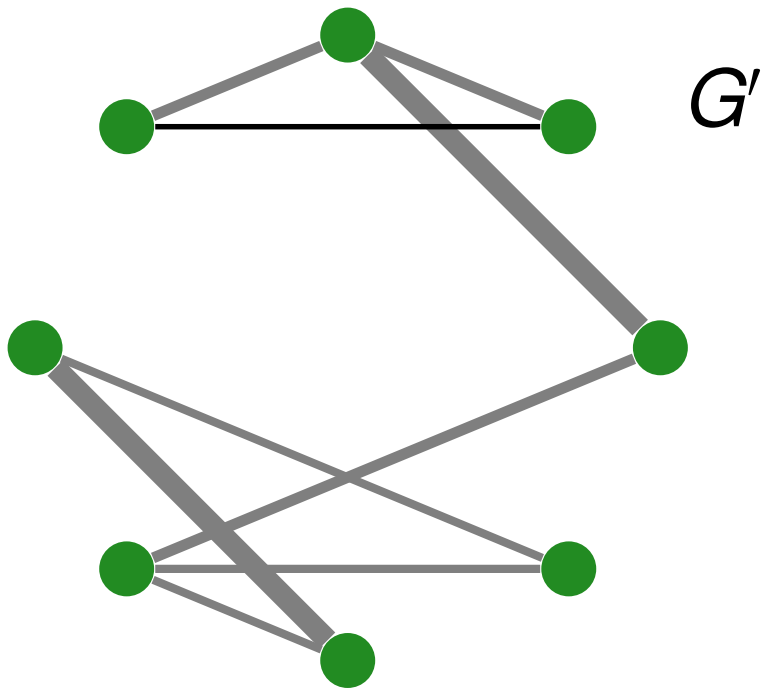

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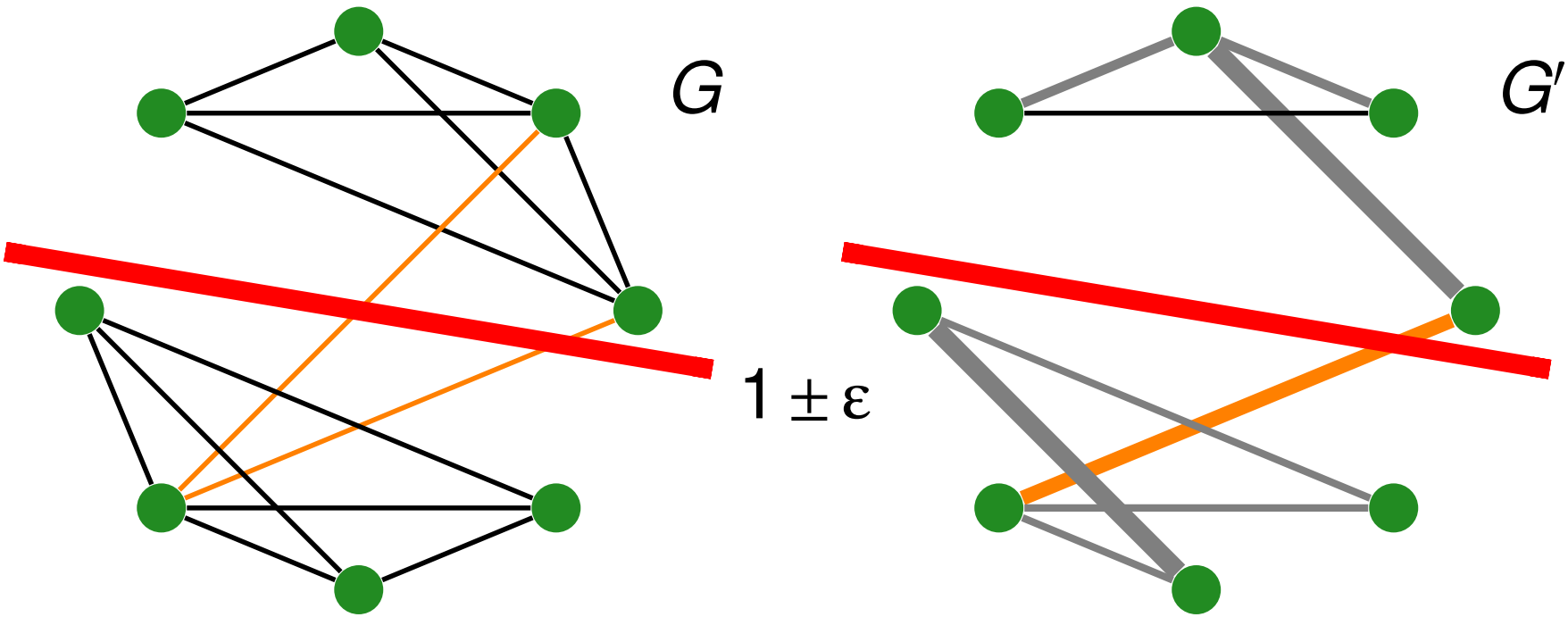

1. Spectral sparsification
2. Dynamic streaming and linear sketching
3. Spectral sparsification using small sketches
4. Fast decoding

1. **Spectral sparsification**
2. Dynamic streaming and linear sketching
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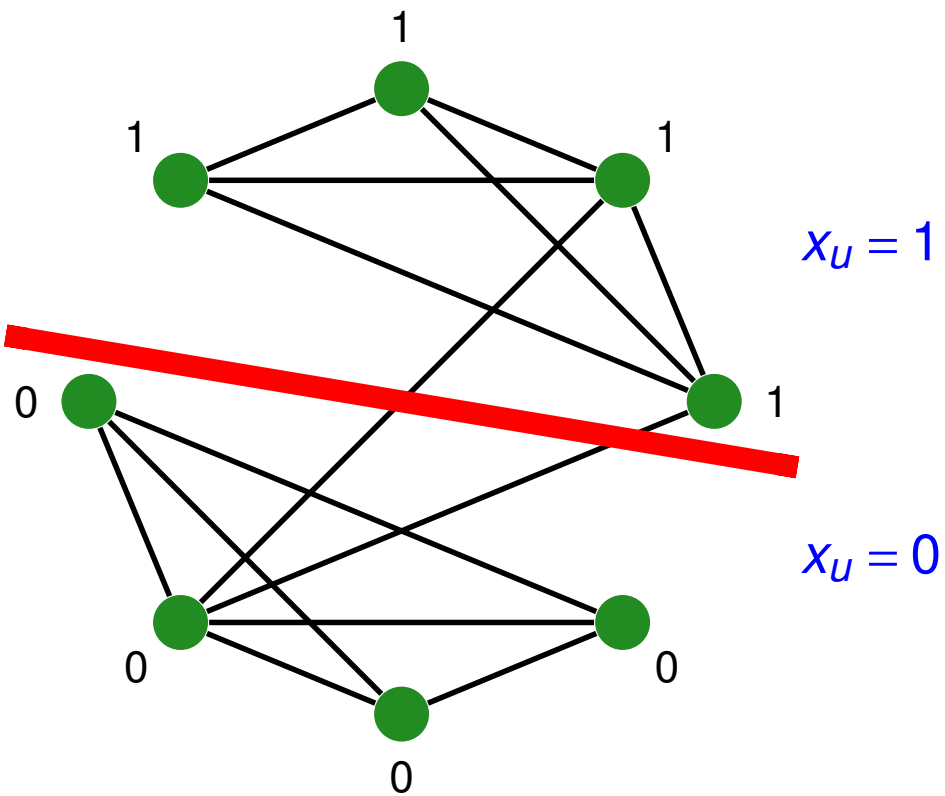
4. Fast decoding

Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2$$

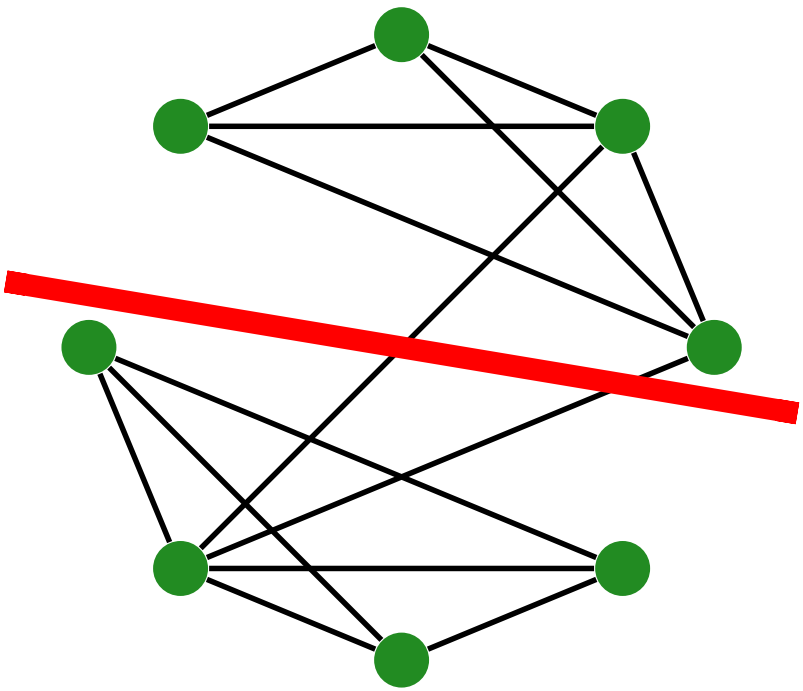


$x_u = 1$

$x_u = 0$

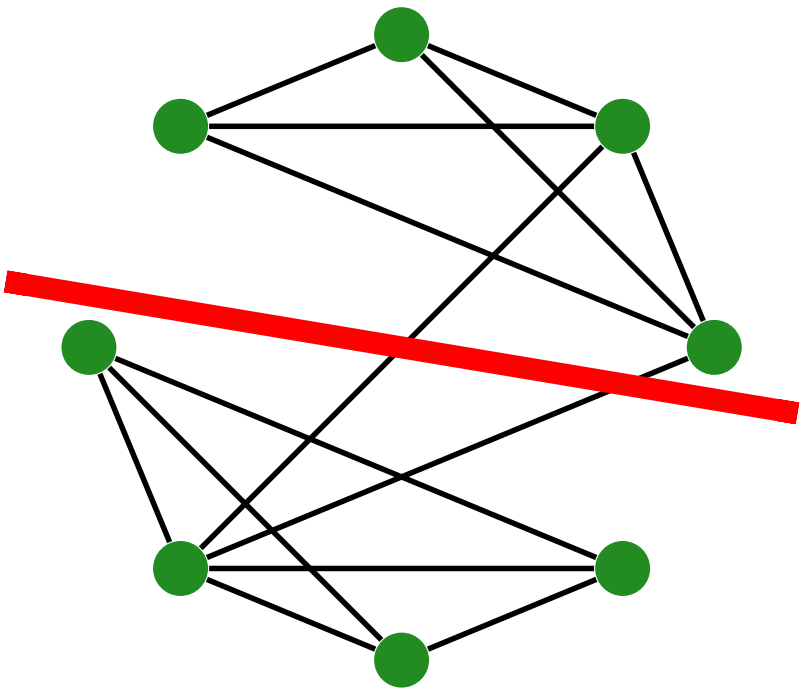
Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2$$



Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2 = ||z||^2$$



$$z = \begin{bmatrix} 0 \\ x_1 - x_2 \\ 0 \\ x_4 - x_3 \\ 0 \\ x_4 - x_5 \\ \vdots \end{bmatrix}$$

Sparsification

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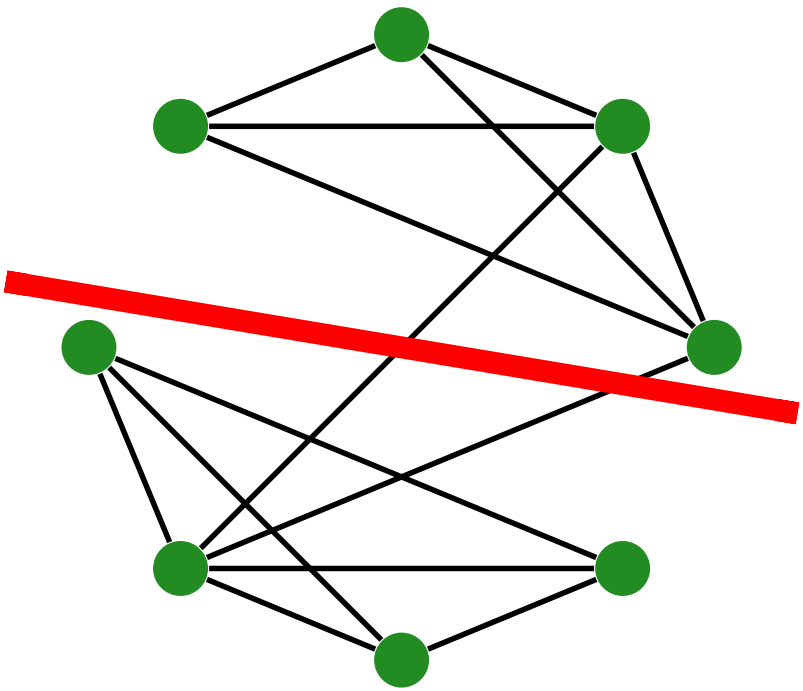

$$z = \begin{bmatrix} 0 \\ x_1 - x_2 \\ 0 \\ x_4 - x_3 \\ 0 \\ x_4 - x_5 \\ \vdots \end{bmatrix} = Bx$$

Sparsification

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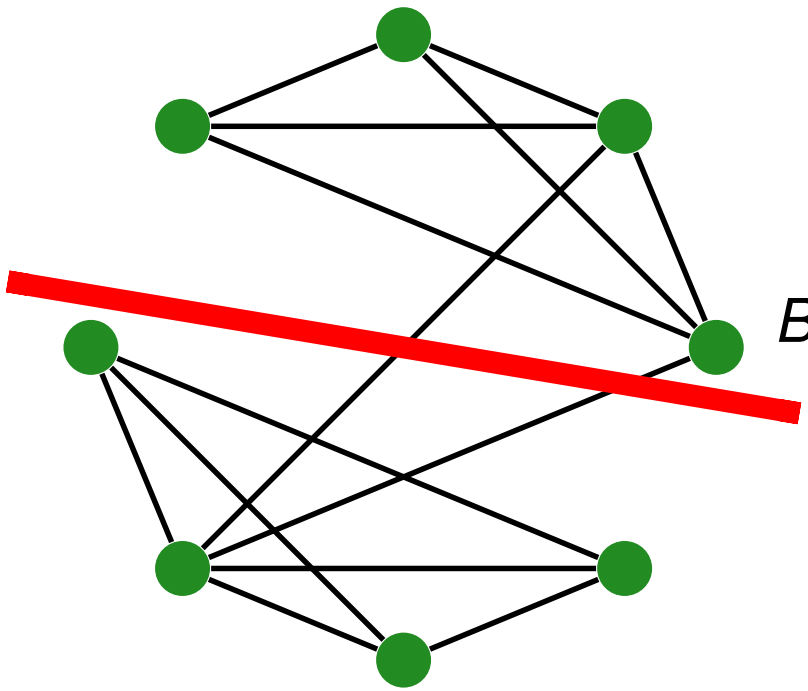

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$\binom{n}{2} \times n$

Sparsification

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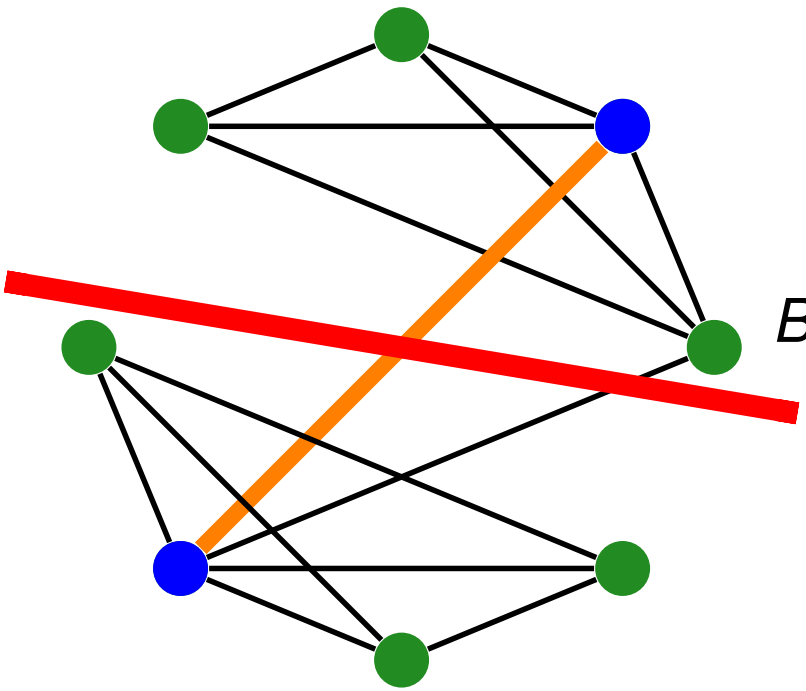

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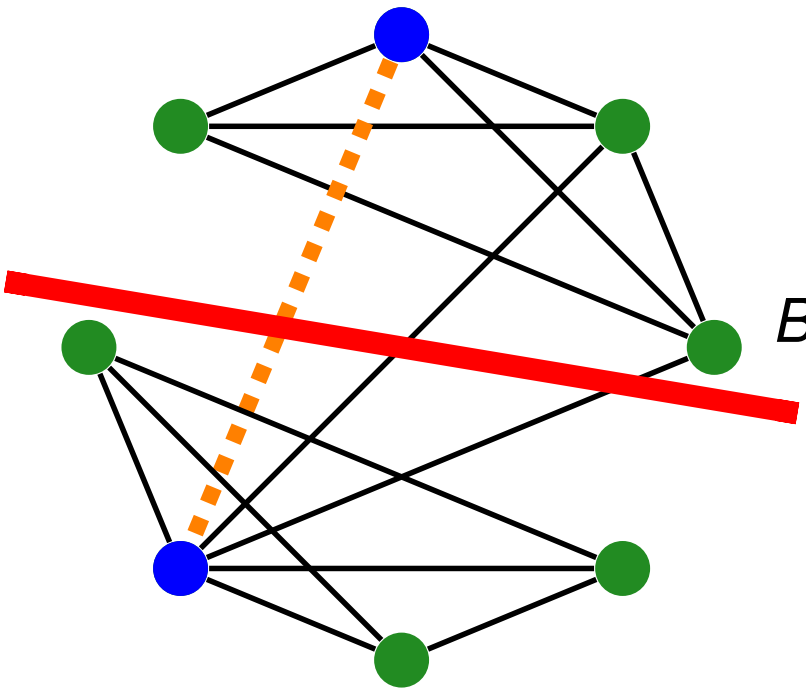

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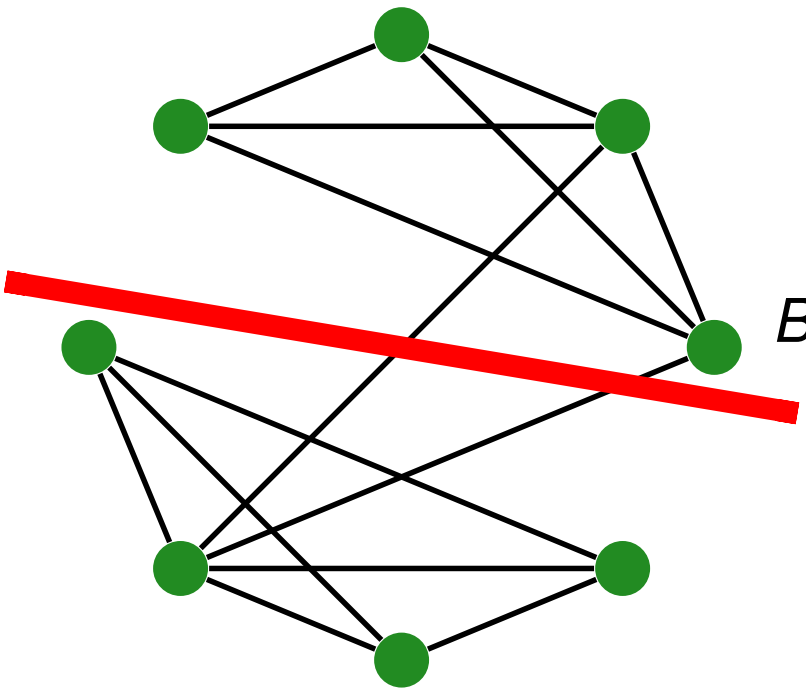
Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2 = ||Bx||^2 = x^T B^T Bx$$

$L = \mathbf{B}^T\mathbf{B}$ is the Laplacian of G



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$\binom{n}{2} \times n$

Definition (Karger'94, Cut sparsifiers)

G' is an ε-cut sparsifier of G if

$$(1-\varepsilon)x^T Lx \leq x^T L'x \leq (1+\varepsilon)x^T Lx$$

for all $x \in \{0,1\}^V$ (all cuts).

Theorem (Karger'94, Benczur-Karger'96)

For any G there exists an ε-cut sparsifier G' with $O(\frac{1}{\varepsilon^2} n \log n)$ edges, and it can be constructed in $\tilde{O}(m)$ time.

Definition (Spielman-Teng'04, Spectral sparsifiers)

G' is an ε-spectral sparsifier of G if

$$(1-\varepsilon)x^T Lx \leq x^T L'x \leq (1+\varepsilon)x^T Lx$$

for all ~~$x \in \{0,1\}^V$ (all cuts).~~ all $x \in \mathbb{R}^V$.

Equivalently, $(1-\varepsilon)L \prec L' \prec (1+\varepsilon)L$

Theorem (Spielman-Teng'04, Spielman-Srivastava'09)

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For any G there exists an ε-spectral sparsifier G' with $O(\frac{1}{\varepsilon^2} n \log n)$ edges, and it can be constructed in $\tilde{O}(m)$ time.

Central role in numerical linear algebra, combinatorial optimization etc

Constructing spectral sparsifiers

Theorem (Spielman-Srivastava'09)

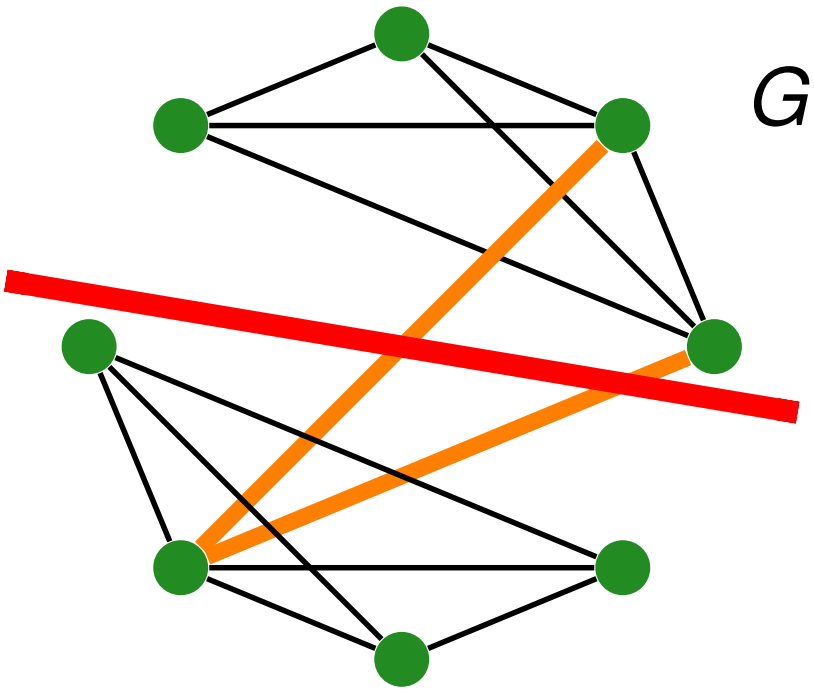
Let $G = (V, E)$ be an undirected graph. Let G' be obtained by including every edge $e \in E$ independently with probability proportional to its effective resistance:

$$p_e \geq \min\left\{1, \frac{C \log n}{\varepsilon^2} R_{uv}\right\}.$$

Assigning weight $1/p_e$ if sampled. Then $(1-\varepsilon)L \prec L' \prec (1+\varepsilon)L$ whp.

Sample edges according to a measure of importance, assign weights to make estimate unbiased

Sparsification



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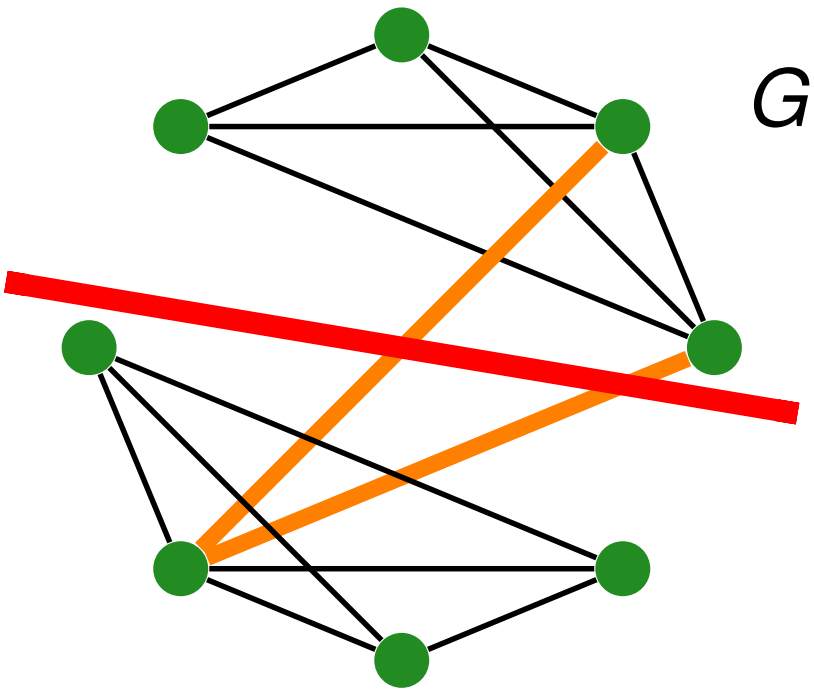

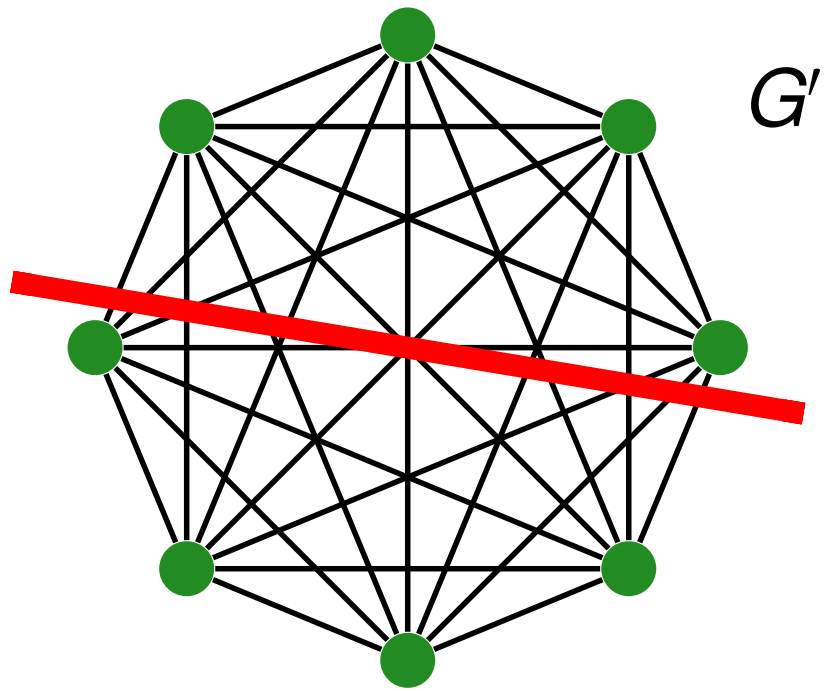

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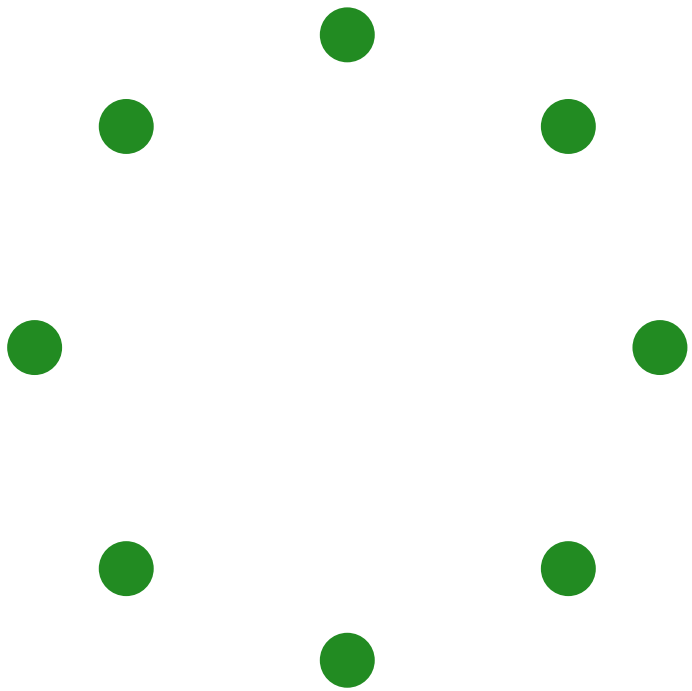
Streaming model

- edges of G arrive in an arbitrary order in a stream
- algorithm can only use $\tilde{O}(n)$ space
- several passes over the stream



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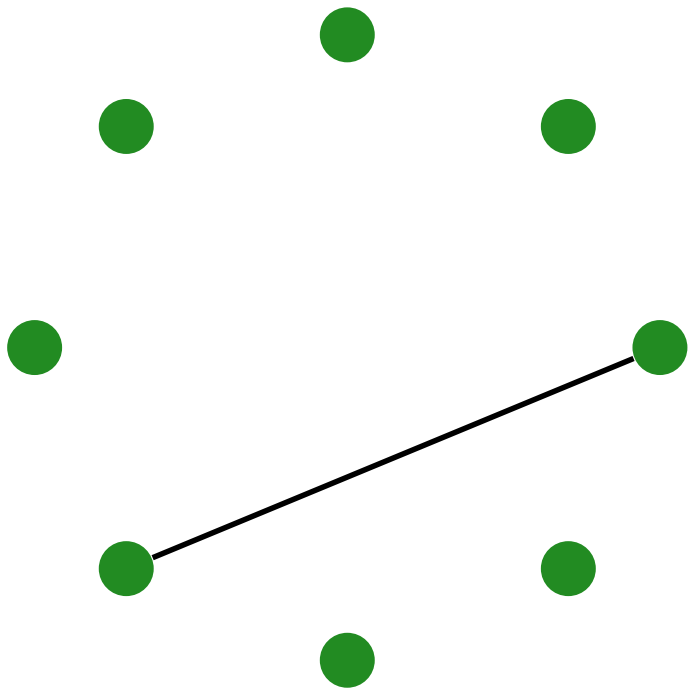

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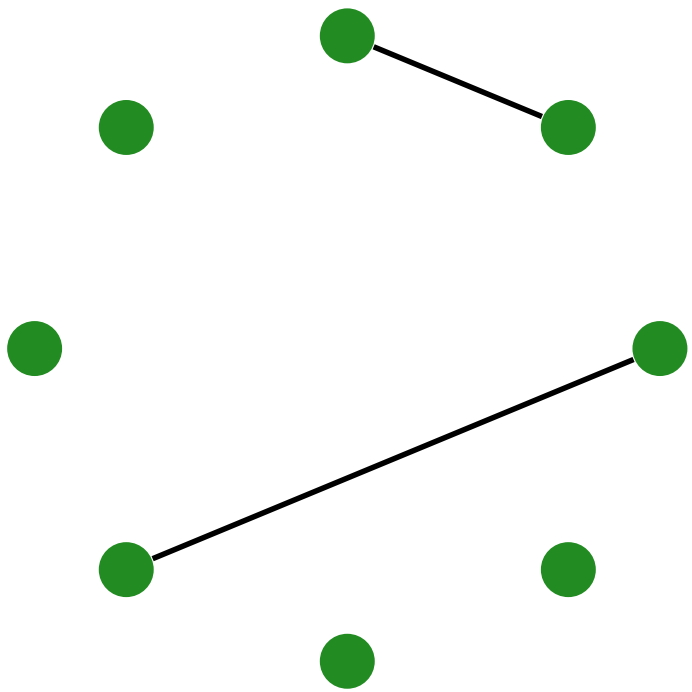

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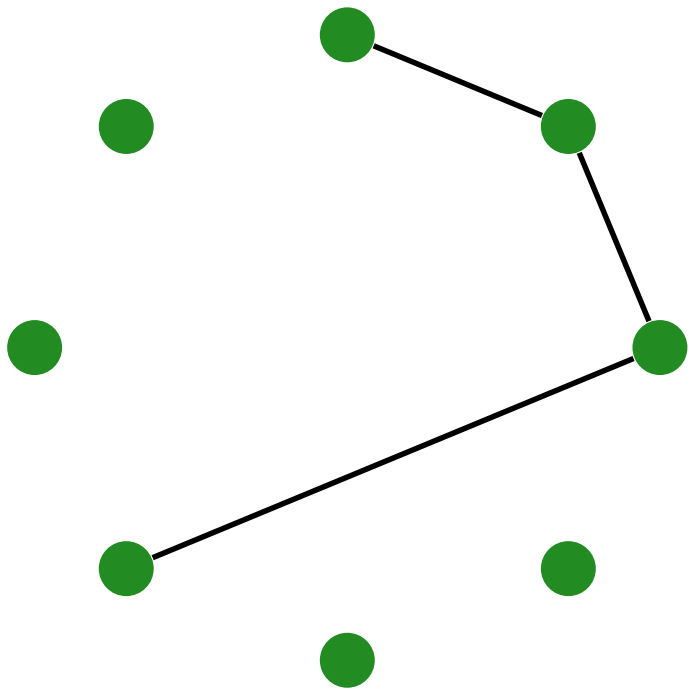

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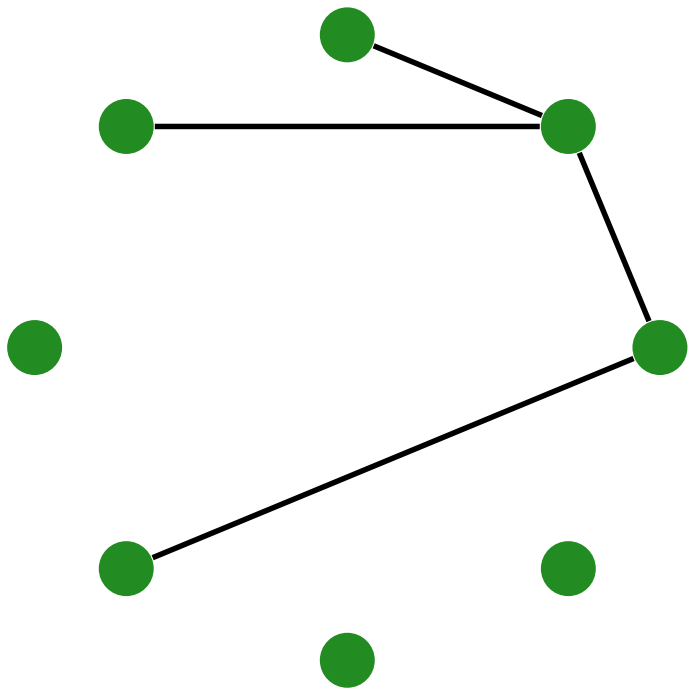

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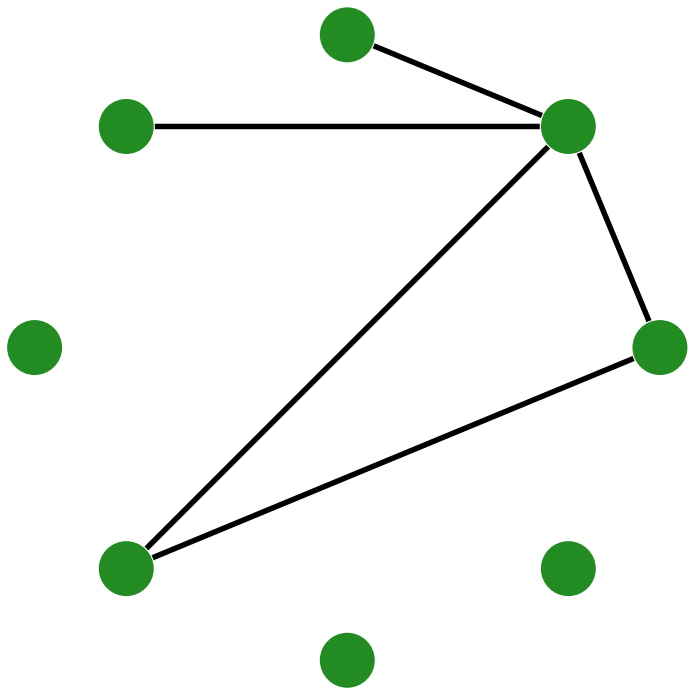

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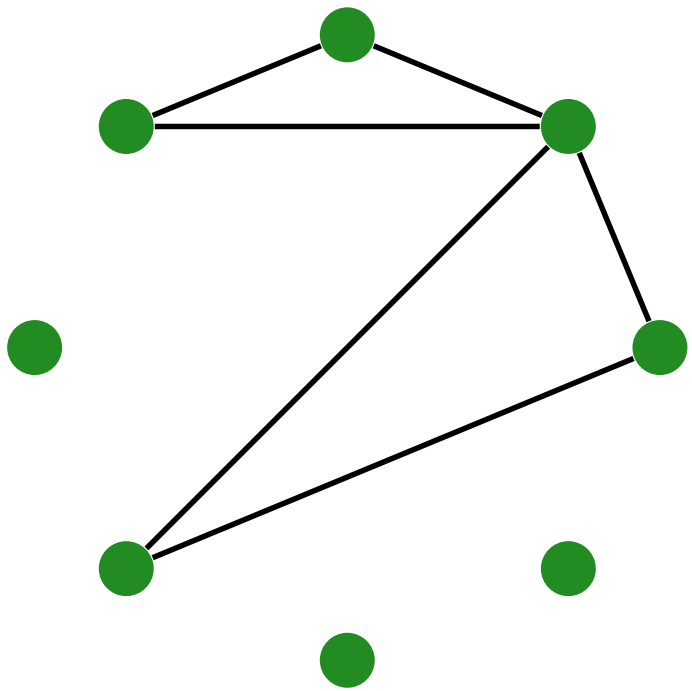

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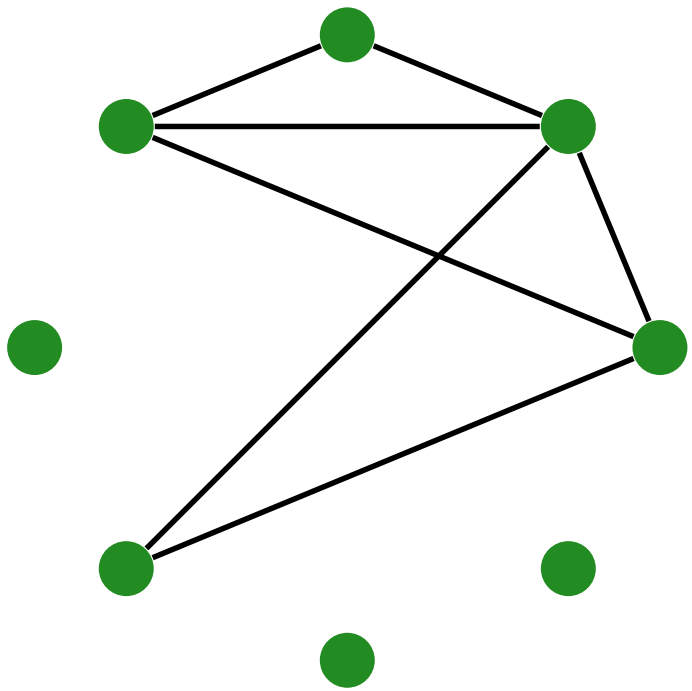

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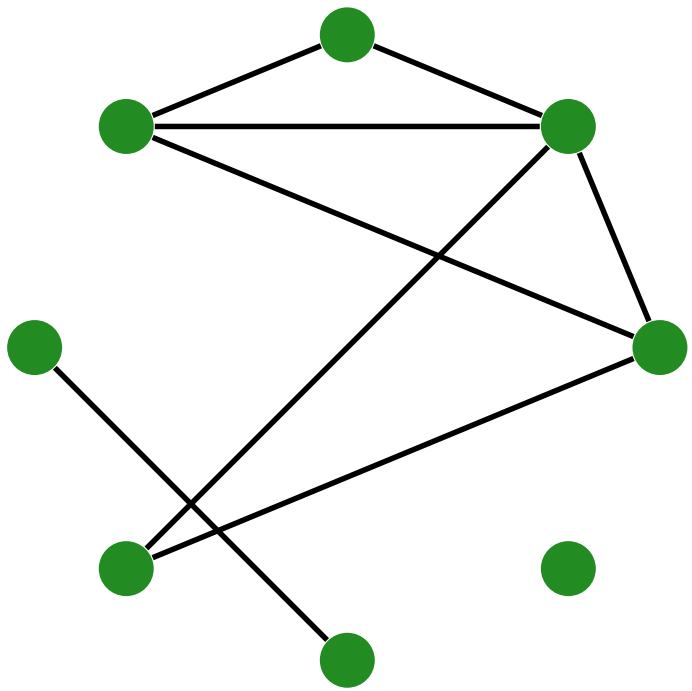

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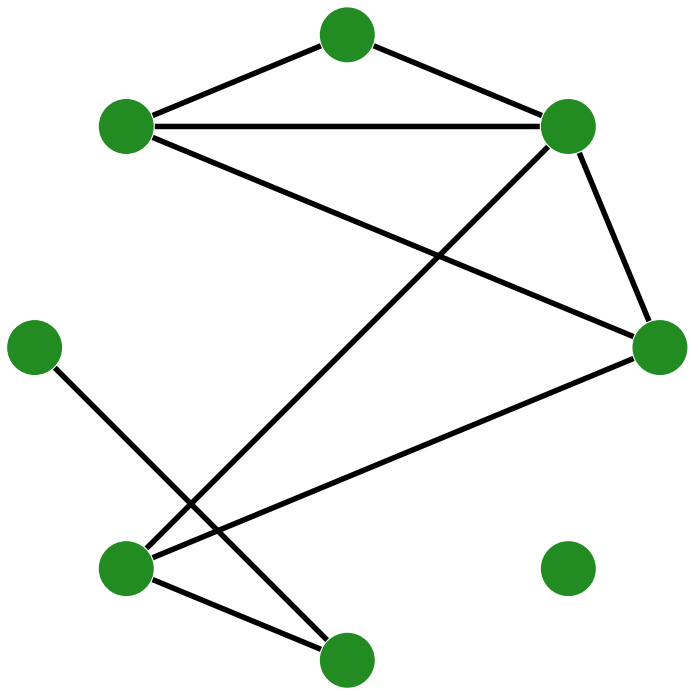

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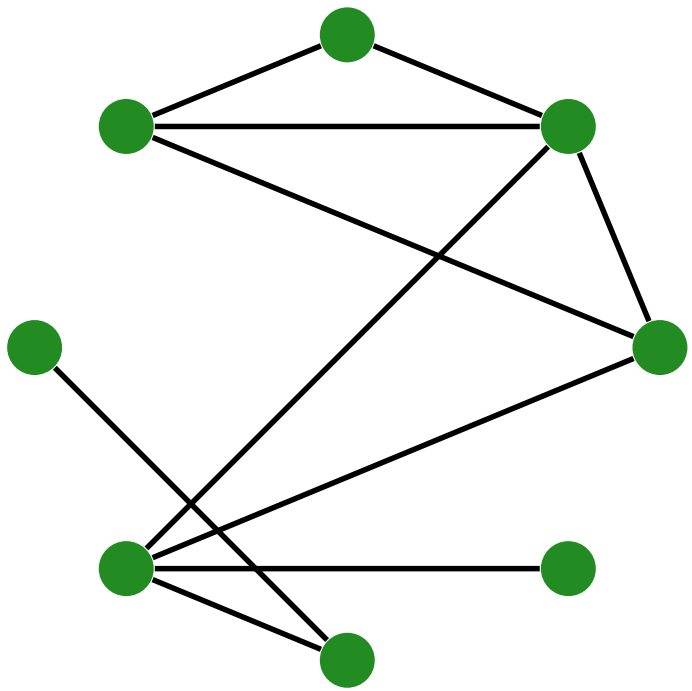

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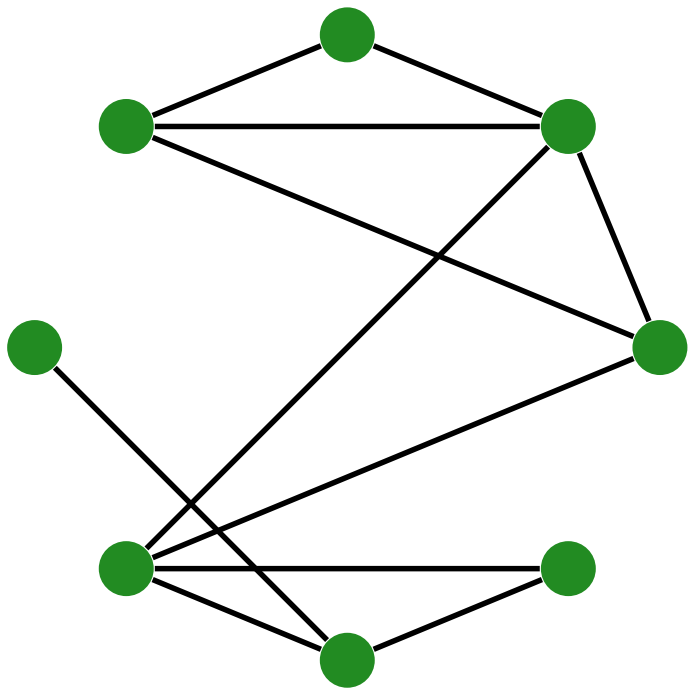

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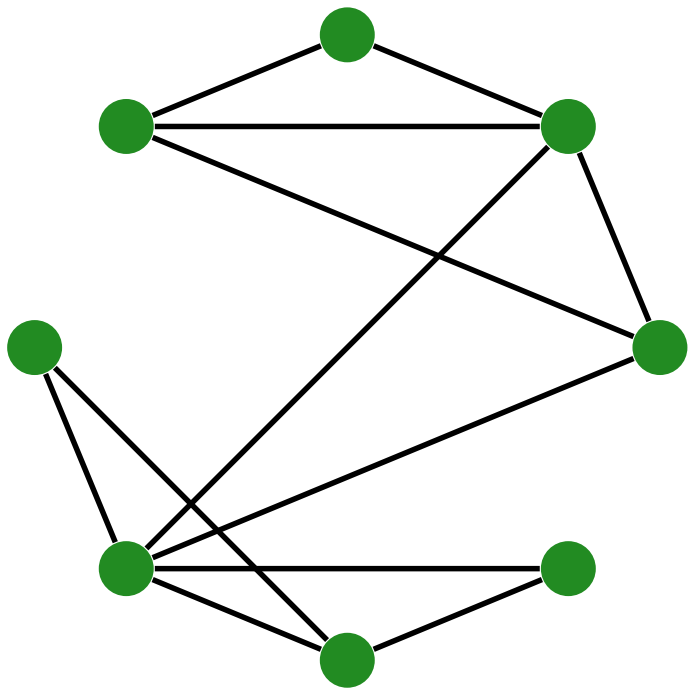

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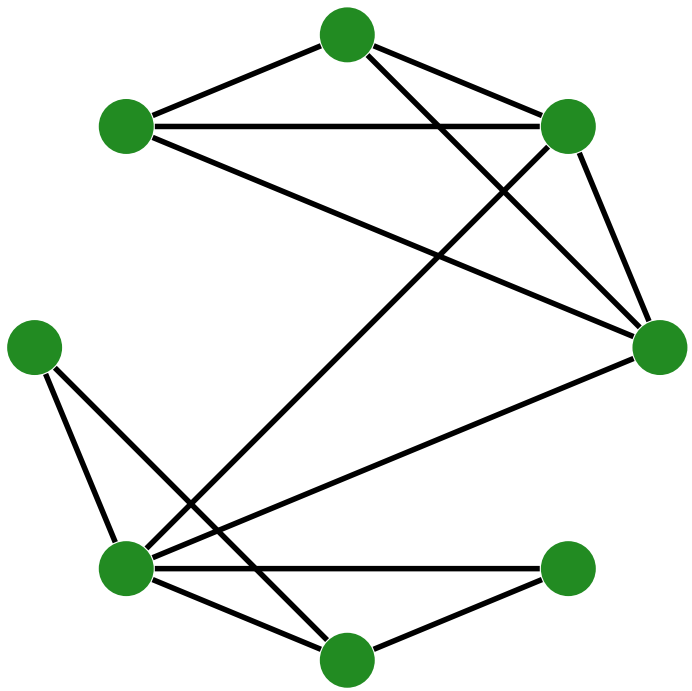

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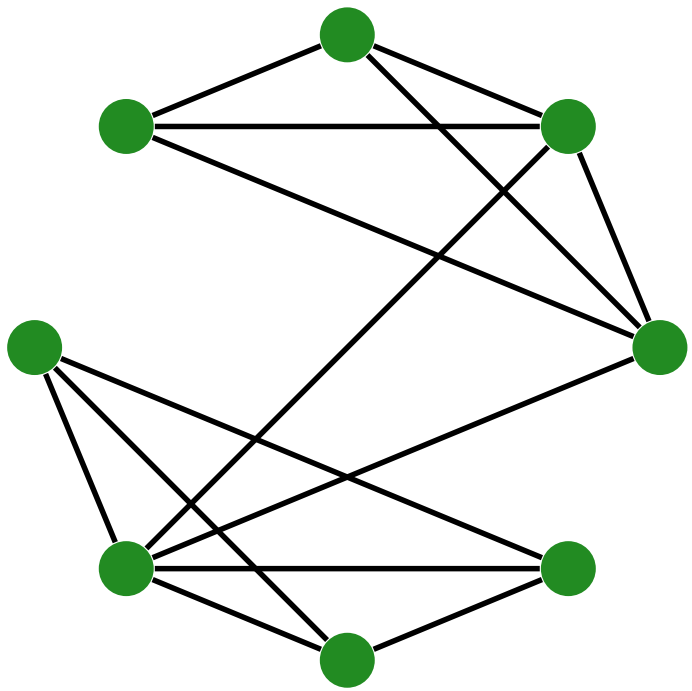

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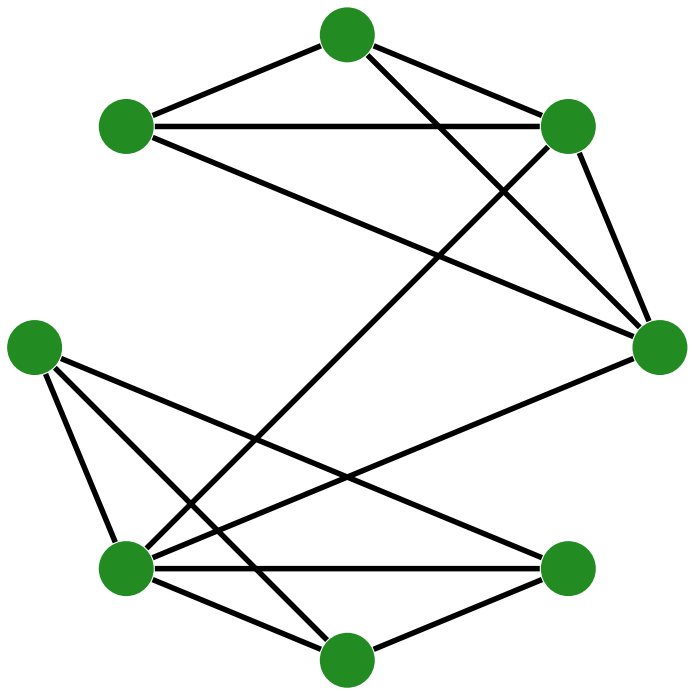

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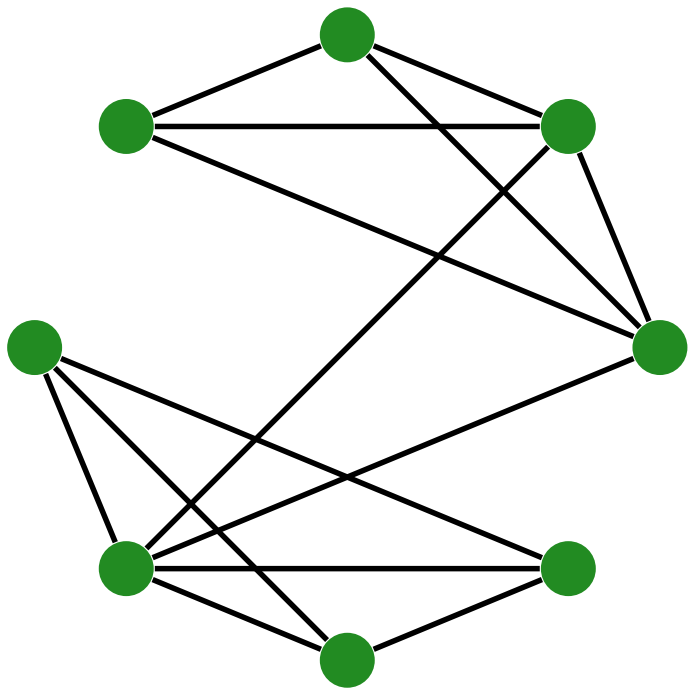

Insertion-only stream

Streaming model

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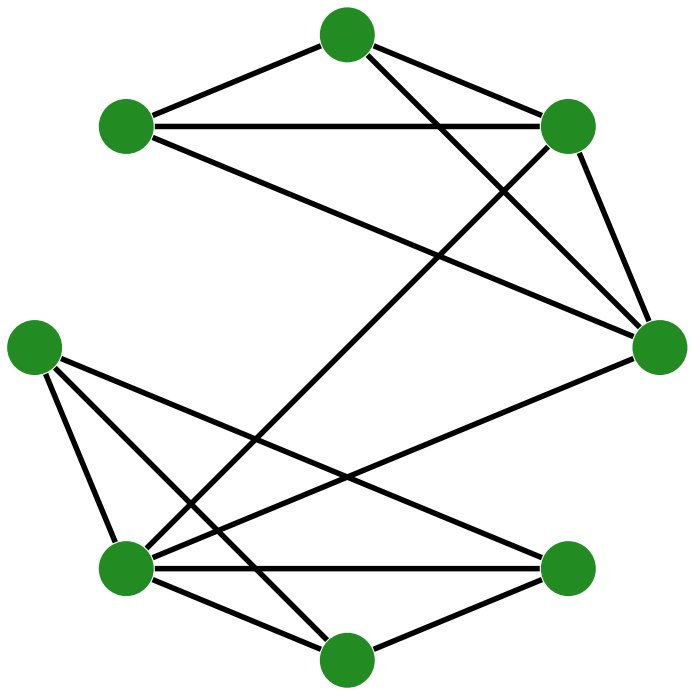
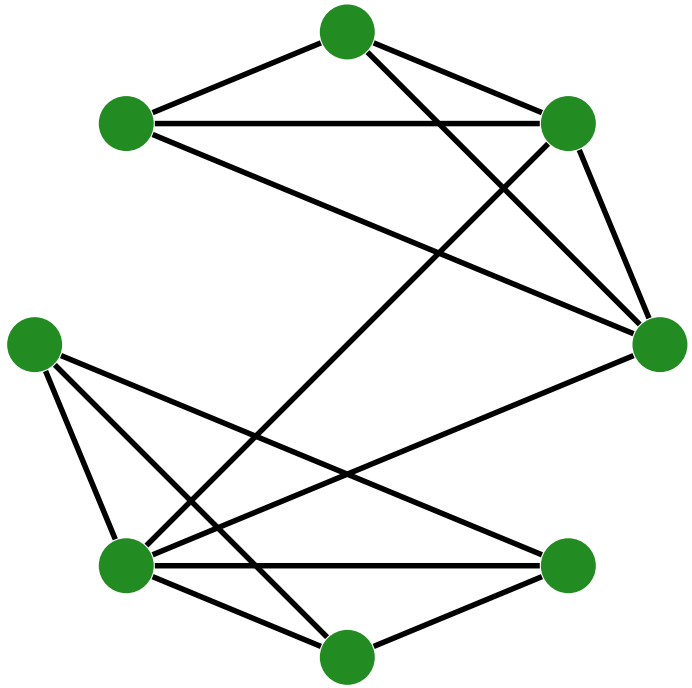


~~Insertion-only~~ stream

What if we have deletions?



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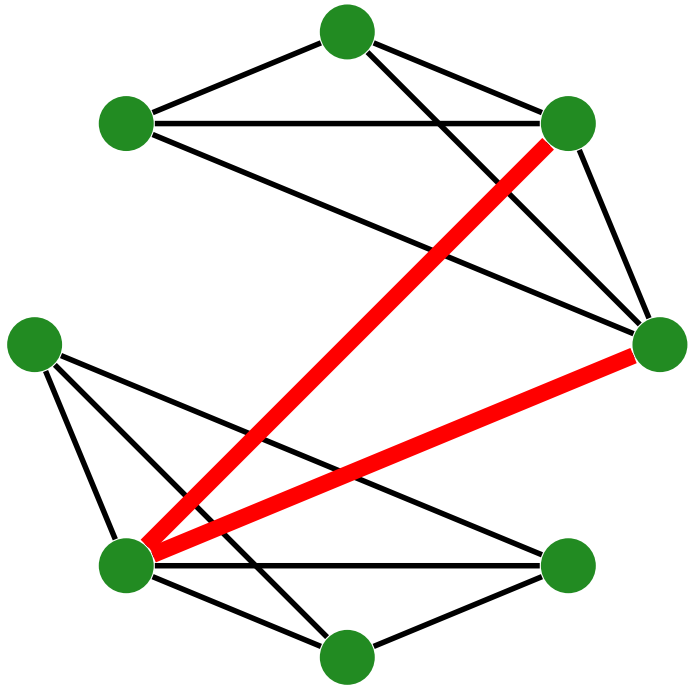

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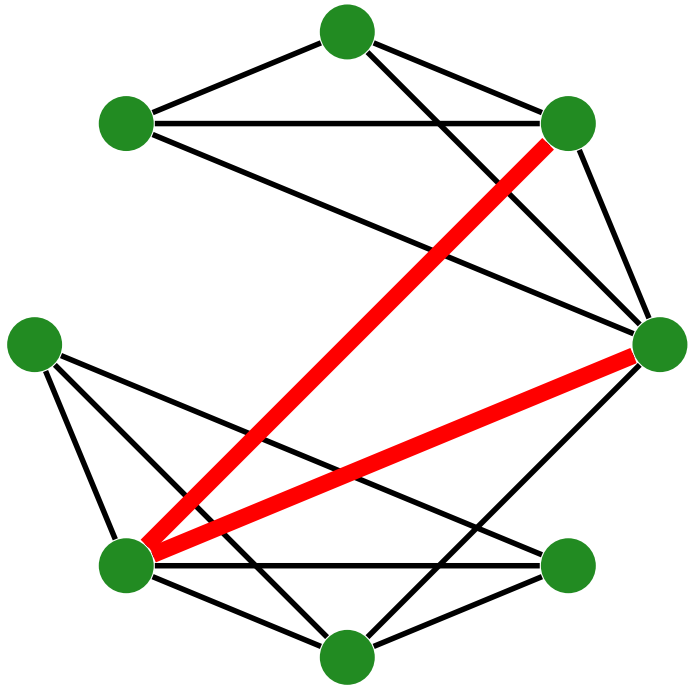

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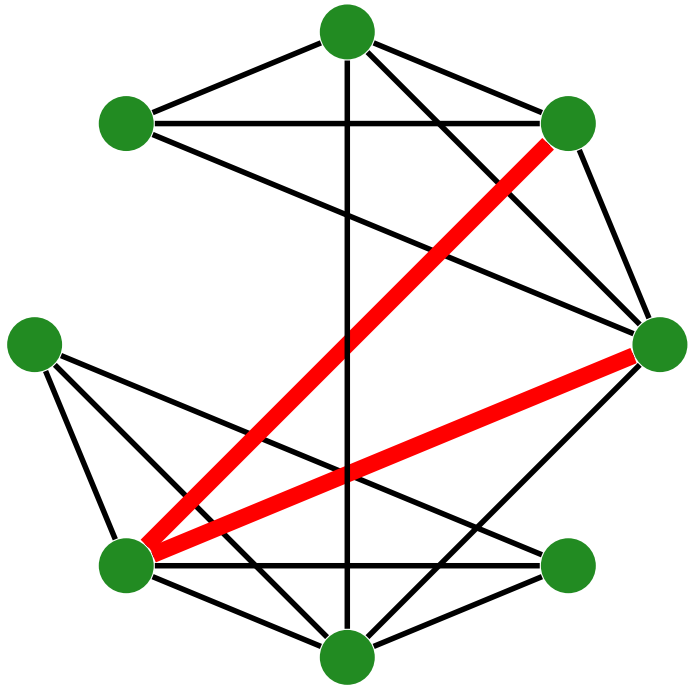

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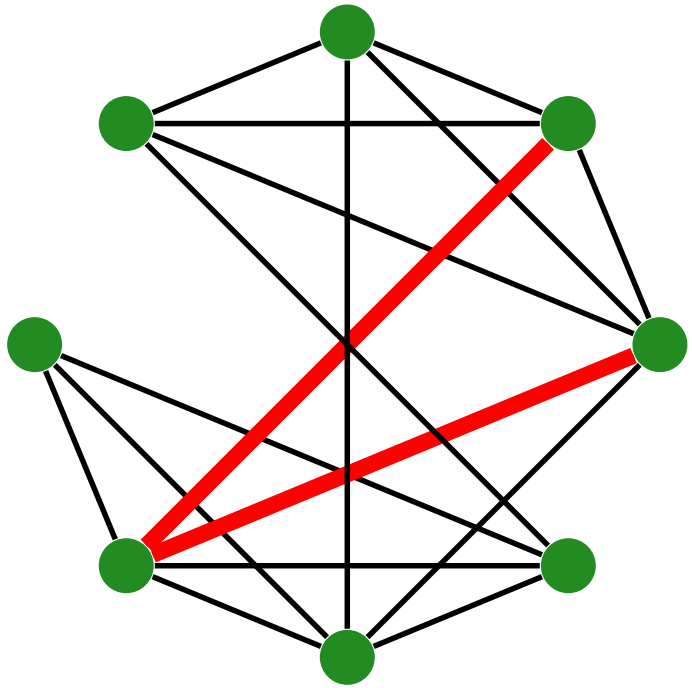

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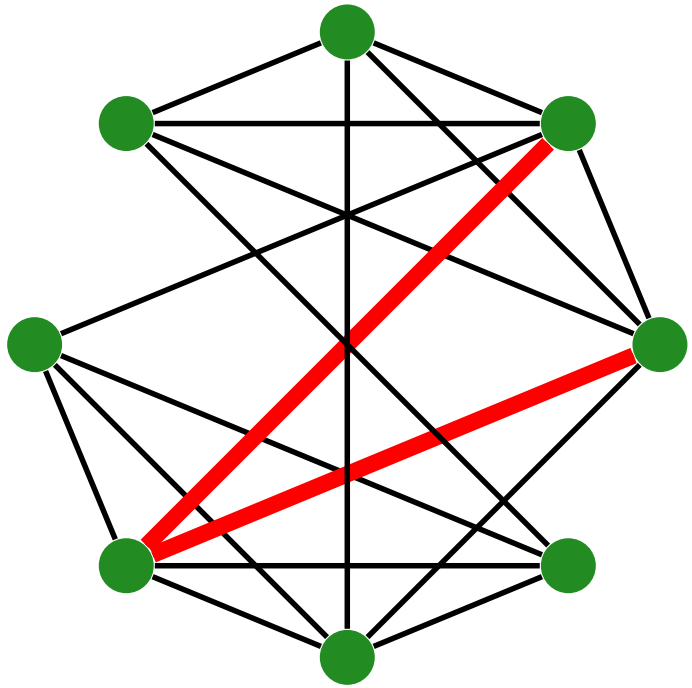

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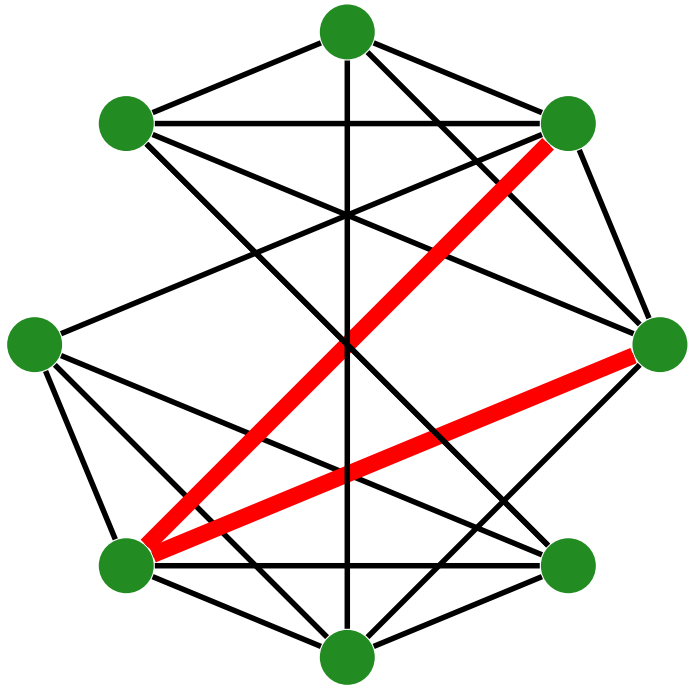

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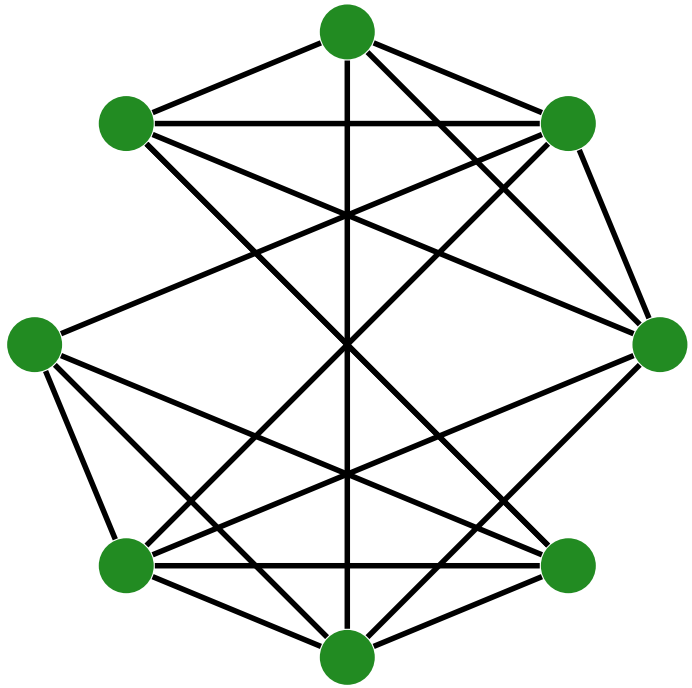

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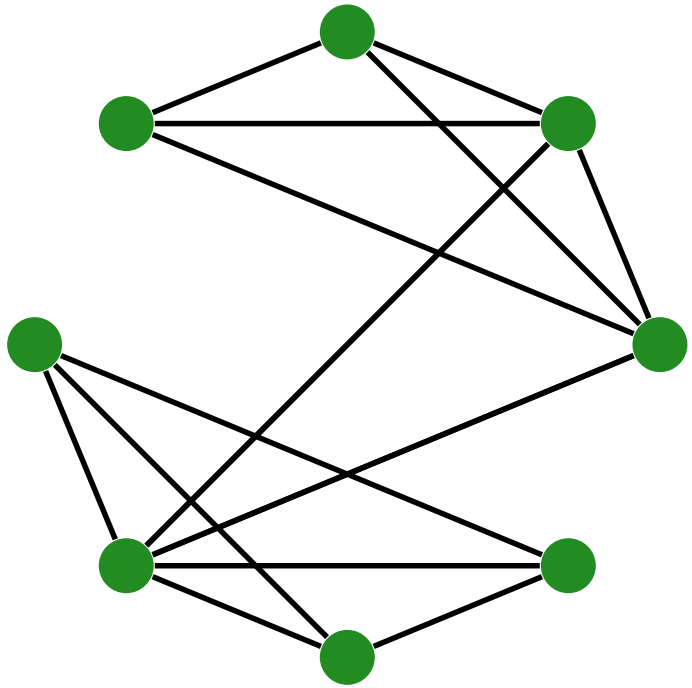

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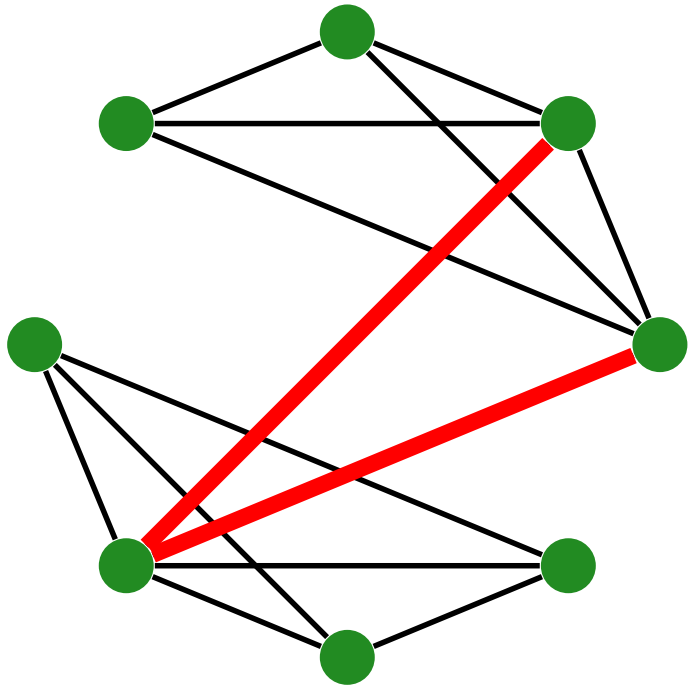
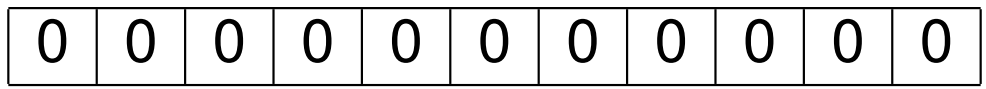


Very different algorithms are needed...

Linear sketching

Classical data stream application: approximating frequency moments.



Goal: approximate $||x||_2^2 = \sum_i x_i^2$ using $\ll n$ space

Linear sketching

Classical data stream application: approximating frequency moments.

0	0	**1**	0	0	0	0	0	0	0	0

Goal: approximate $||x||_2^2 = \sum_i x_i^2$ using $\ll n$ space

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Classical data stream application: approximating frequency moments.

0	0	1	0	**1**	0	0	0	0	0	0

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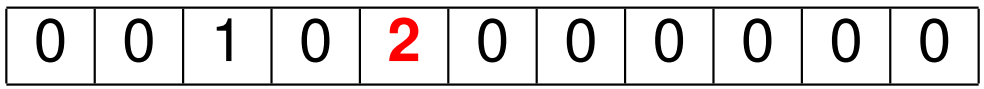

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Linear sketching

Classical data stream application: approximating frequency moments.

0	0	1	0	**2**	0	0	0	0	0	0

Goal: approximate $||x||_2^2 = \sum_i x_i^2$ using $\ll n$ space

Maintain $x^T v_i = 1, \ldots, O(1/\varepsilon^2)$ for random Gaussians $v_i \in \mathbb{R}^n$.
Output average of $(x^T v_i)^2$.

Linear sketching

Classical data stream application: approximating frequency moments.

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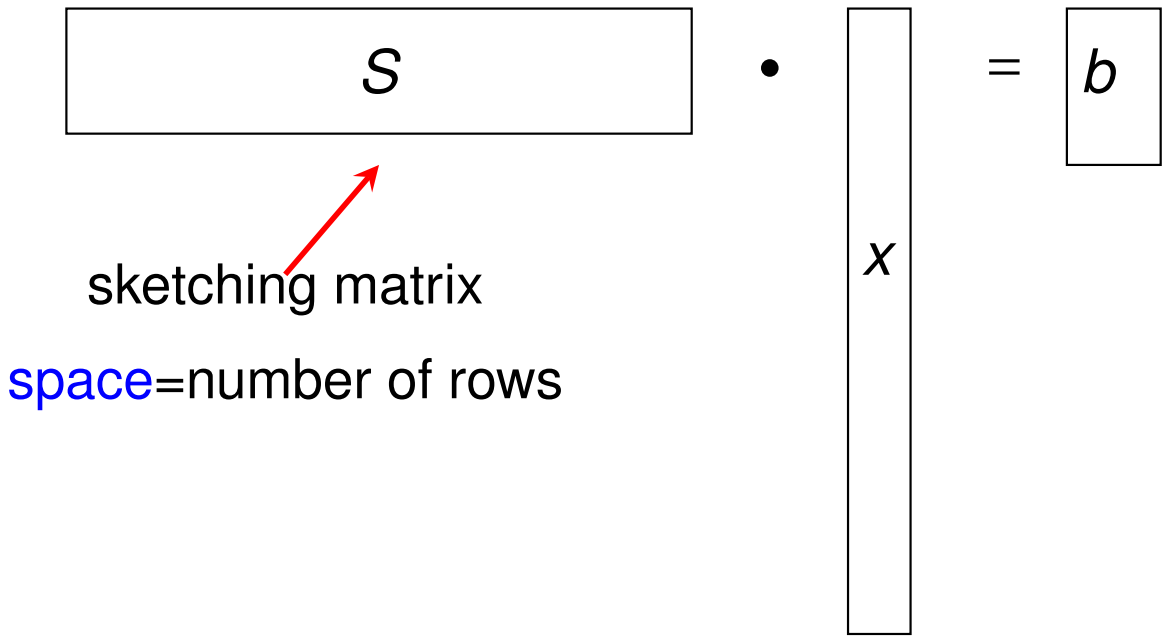
$(1 \pm \varepsilon)$-approximation with $O(\frac{1}{\varepsilon^2} \log n)$ space

Linear sketching

Take (randomized) linear measurements of the input



space=number of rows

Linear sketching

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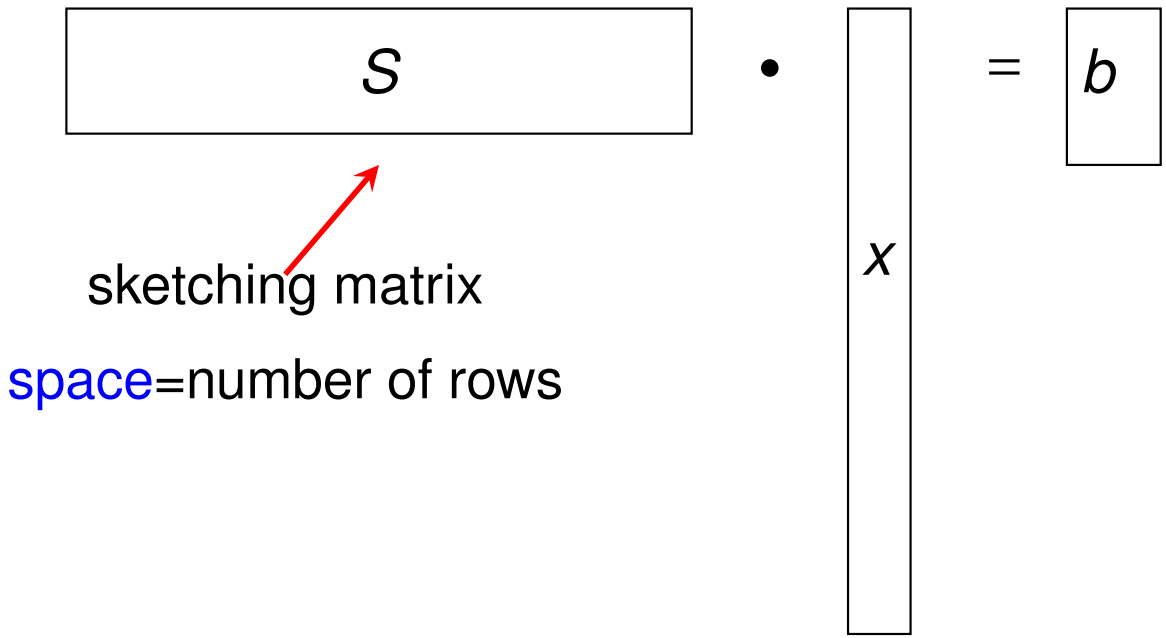

Can get constant approximation using $\log^{O(1)} n$ rows

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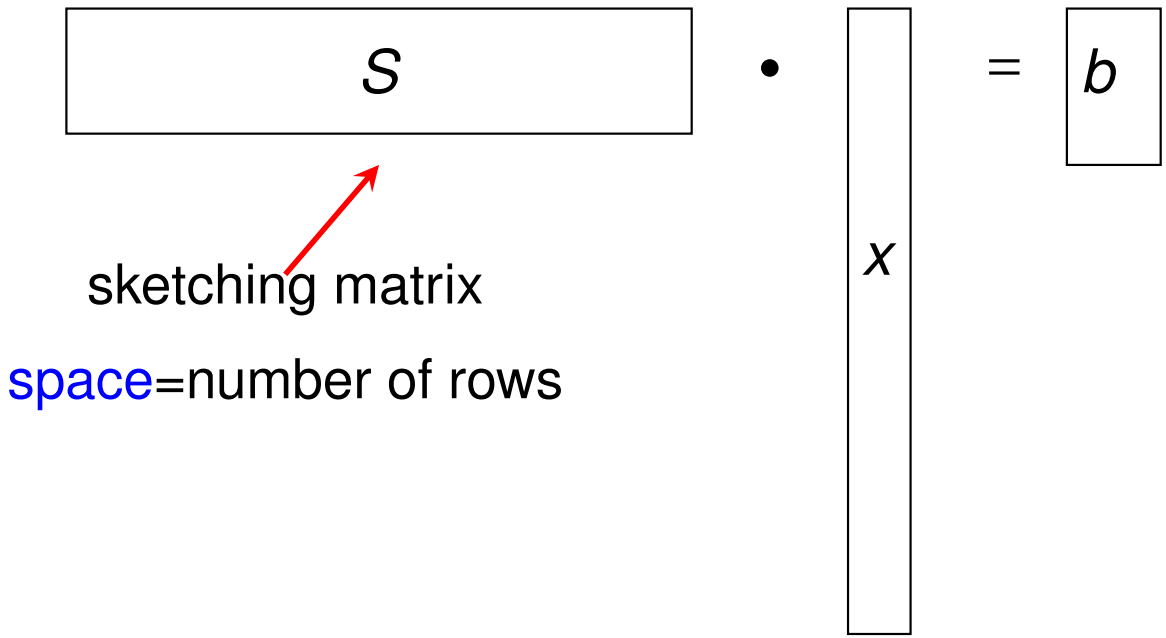

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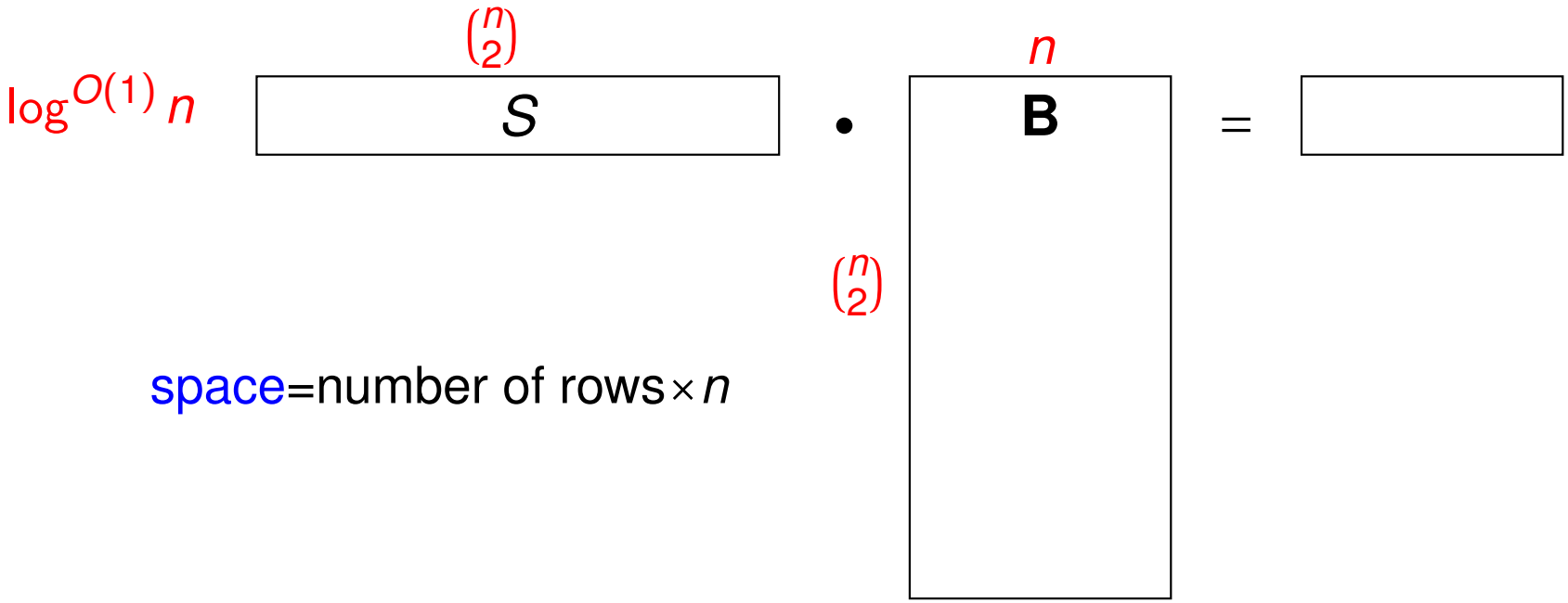
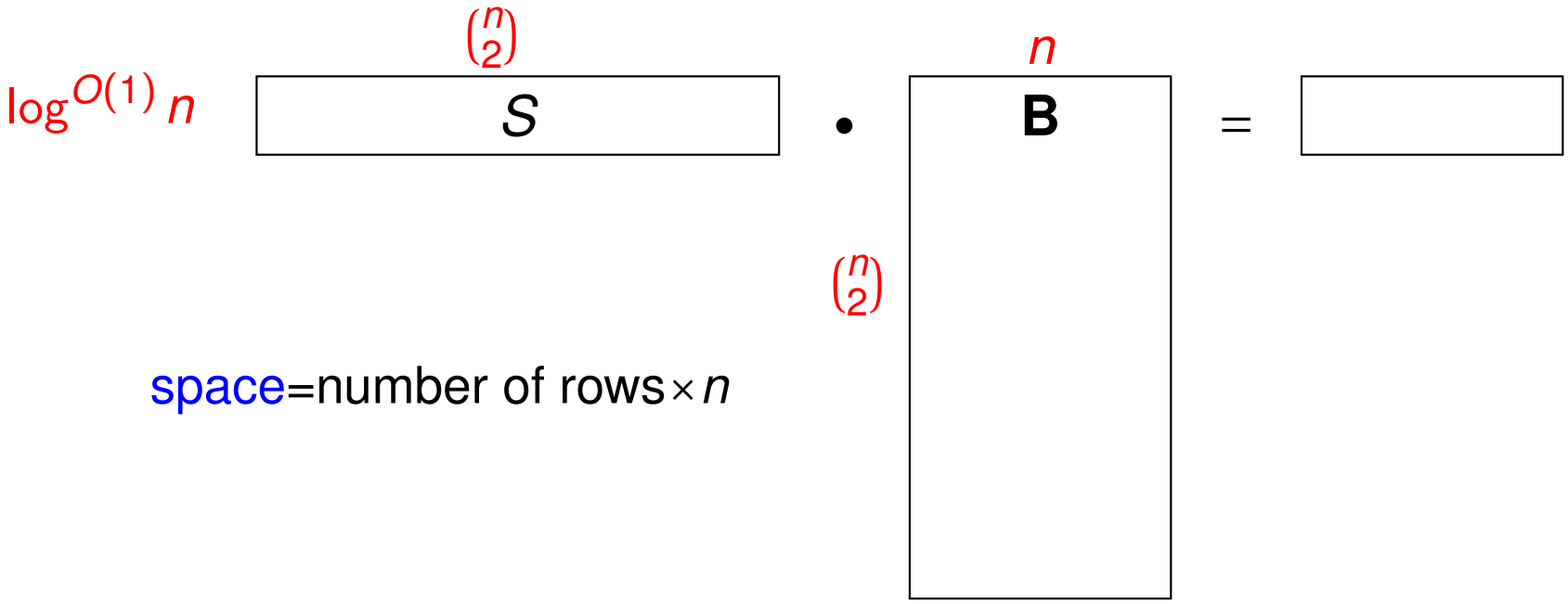
Easy to maintain linear sketches in the (dynamic) streaming model

Ahn-Guha-McGregor'SODA12 – connectivity in $O(n \cdot \log^3 n)$ space.



space=number of rows$\times n$

Ahn-Guha-McGregor'SODA12 – connectivity in $O(n \cdot \log^3 n)$ space.



Every vertex sketches its own neighborhood independently – single round distributed algorithms

Reconstruct edges of a sparsifier quickly from a small sketch?

Reconstruct edges of a sparsifier quickly from a small sketch?

Theorem (K.-Nouri-Sidford-Tardos'19)

Can recover a spectral sparsifier from an oblivious sketch of size $\tilde{O}(n)$ space in time $\tilde{O}(n)$.

Known before:

$\tilde{O}(n)$ space and $\Omega(n^2)$ recovery time
K.-Lee-Sidford-Musco-Musco'FOCS'14

$\tilde{O}(n^{5/3})$ space and time Ahn-Guha-McGregor'APPROX'14

1. Spectral sparsification
2. Dynamic streaming and linear sketching
3. Spectral sparsification using small sketches
4. Fast decoding

1. Spectral sparsification
2. Dynamic streaming and linear sketching
3. **Spectral sparsification using small sketches**
4. Fast decoding

Constructing spectral sparsifiers

Theorem (Spielman-Srivastava'09)

Let $G = (V, E)$ be an undirected graph. Let G' be obtained by including every edge $e \in E$ independently with probability proportional to its effective resistance:

$$p_e \geq \min\left\{1, \frac{C \log n}{\varepsilon^2} R_{uv}\right\}.$$

Assign weight $1/p_e$ if sampled. Then $(1-\varepsilon)G \prec G' \prec (1+\varepsilon)G$ whp.

Sample edges according to a measure of importance,
assign weights to make estimate unbiased

Note: edges e with resistance $R_{uv} = \Omega(1/\log n)$ included w.p. 1

Core primitive: recover edges of large resistance

Goal: design a sketch S that allows recovery of high resistance ($\geq 1/\log n$) edges of G given

- $S \cdot \mathbf{B}$ (sketch of edge incidence matrix)

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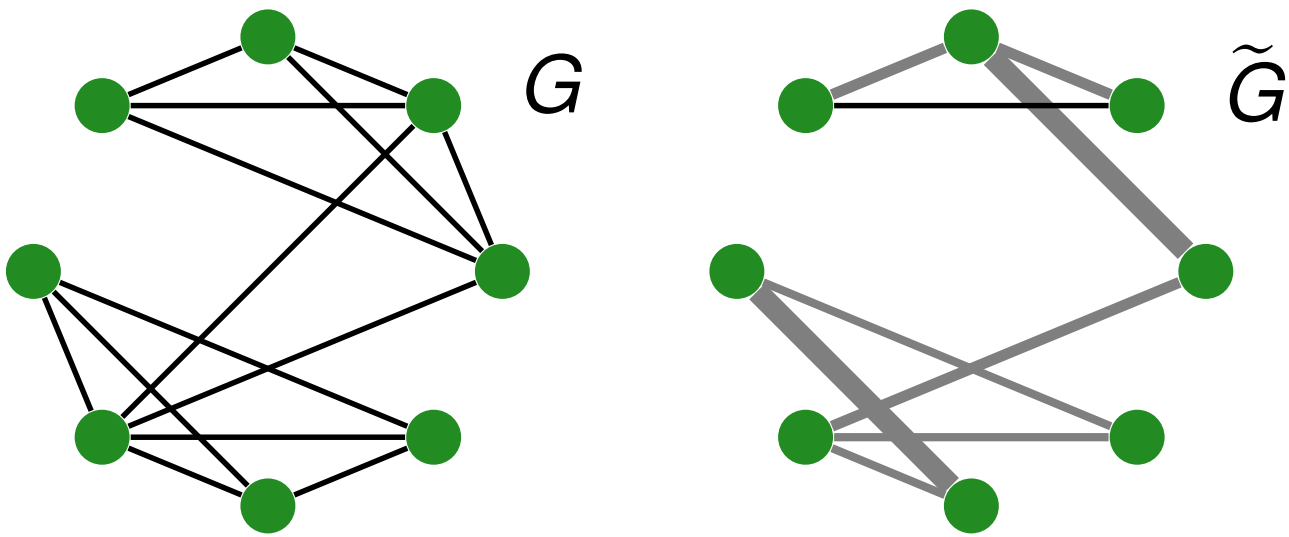
- $S \cdot \mathbf{B}$ (sketch of edge incidence matrix)
- crude constant factor spectral sparsifier $\tilde{G}$

$$\frac{1}{C} \cdot L \prec \tilde{L} \prec L$$



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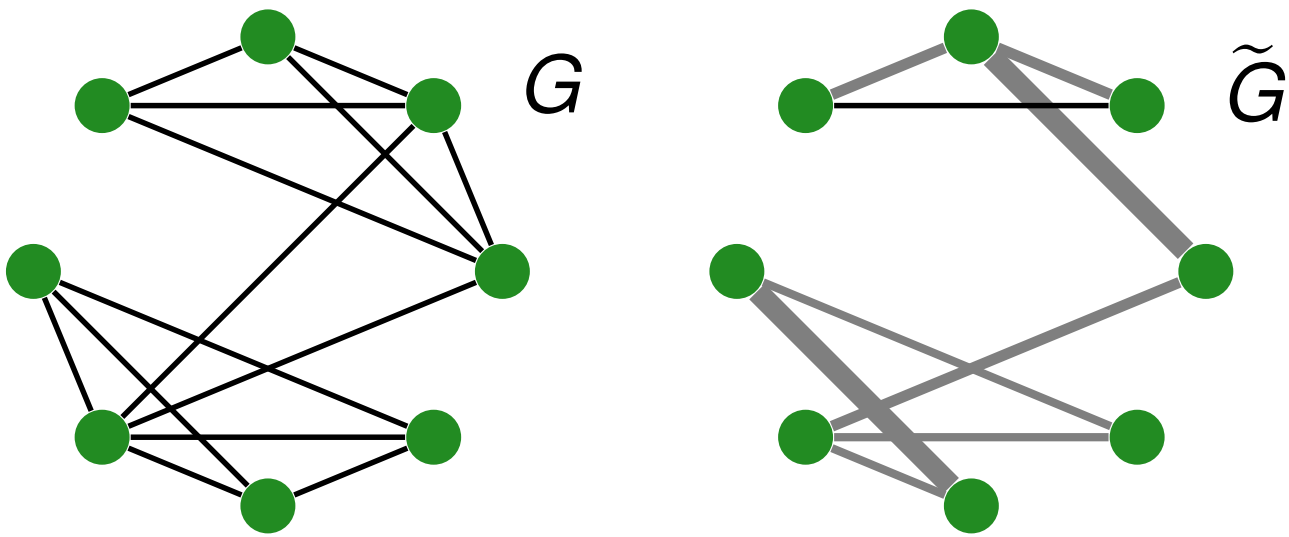
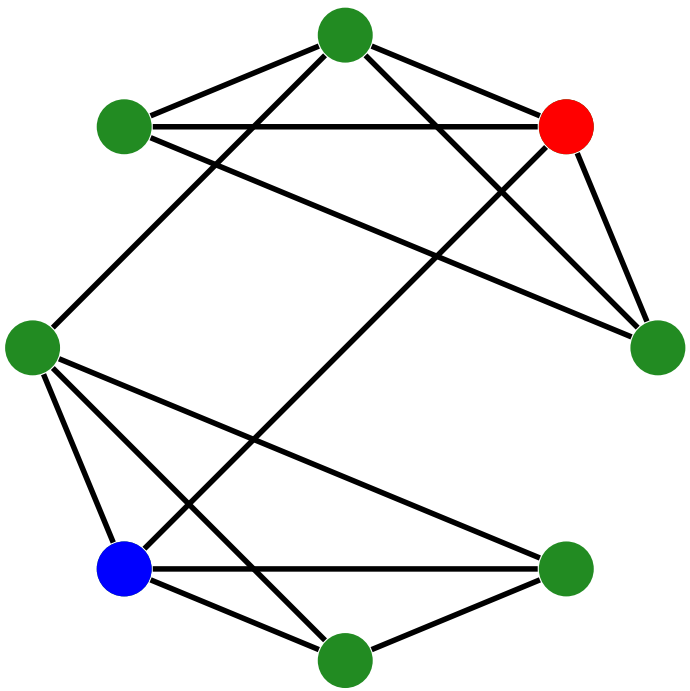


Effective resistance

$$R_{uv} = \mathbf{b}_{uv}^T L^+ \mathbf{b}_{uv}$$

Note: defined for any pair (u, v).



Inject current at u, take out at v.

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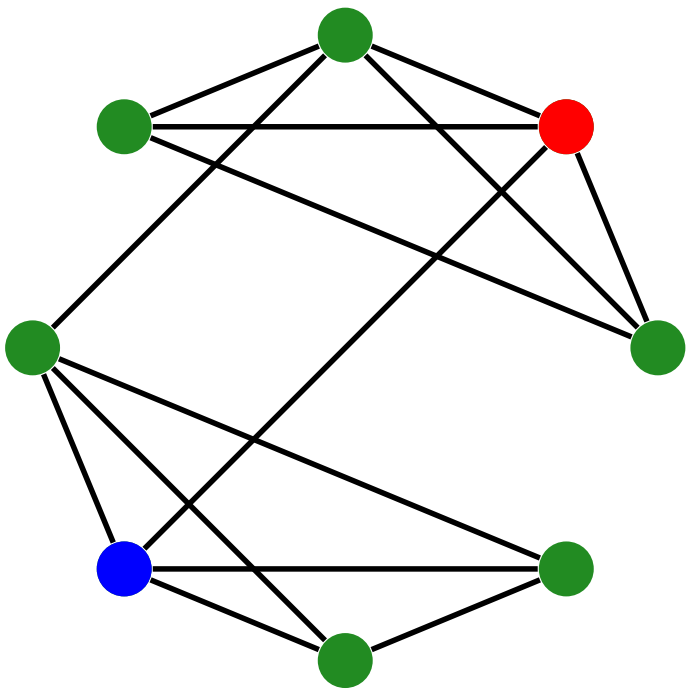

Inject current at u, take out at v.

$\phi = L^+ \mathbf{b}_{uv}$=vertex potentials

$$\mathbf{b}_{uv} = \begin{bmatrix} 0 \\ \mathbf{1} \\ 0 \\ \mathbf{-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Effective resistance

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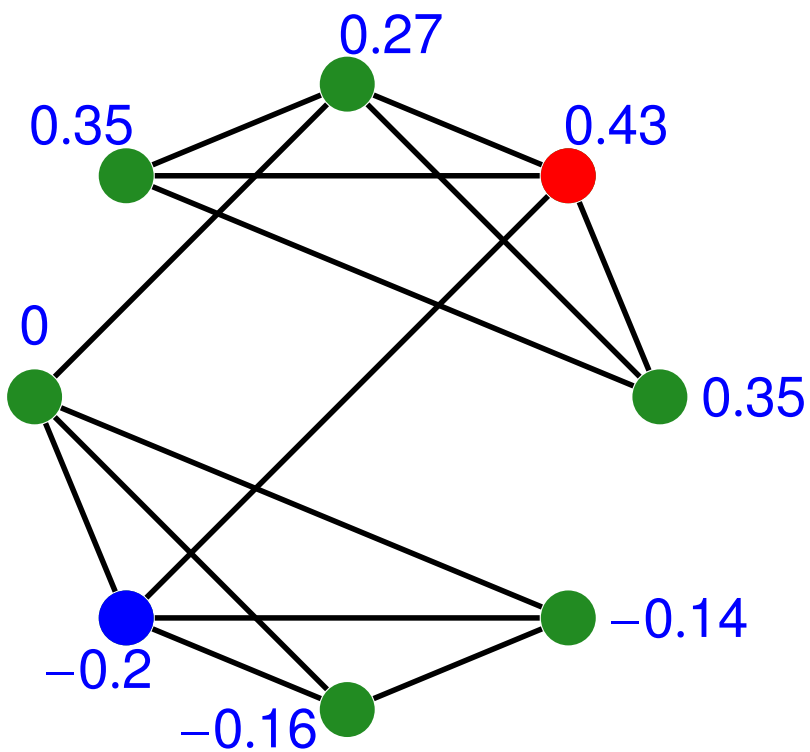

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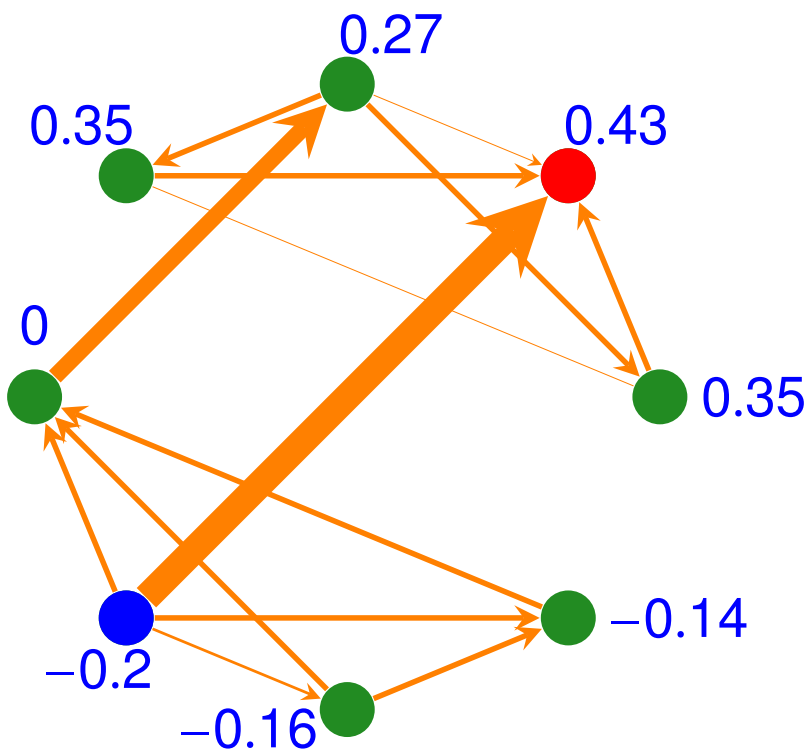

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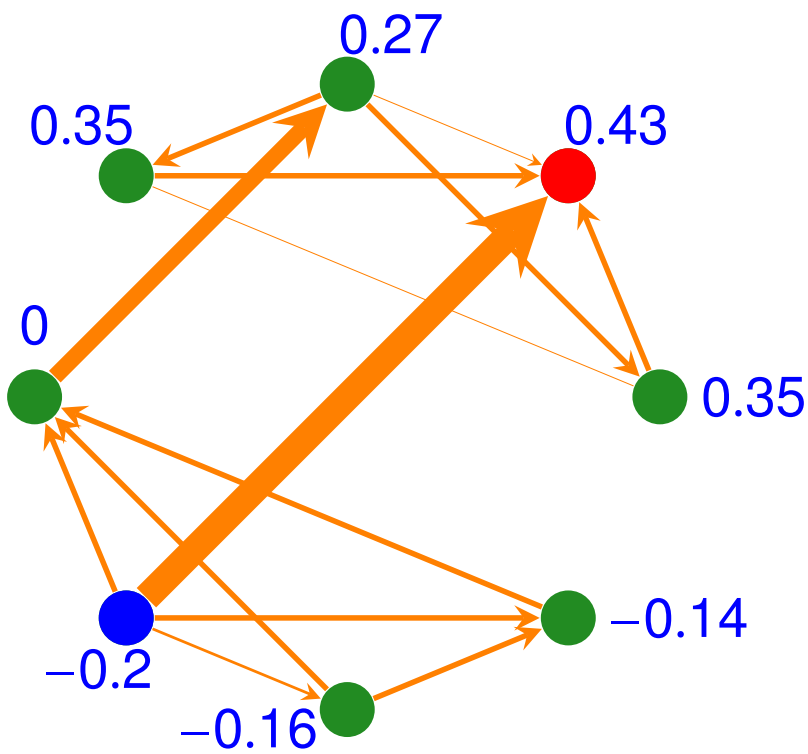


Inject current at u, take out at v.

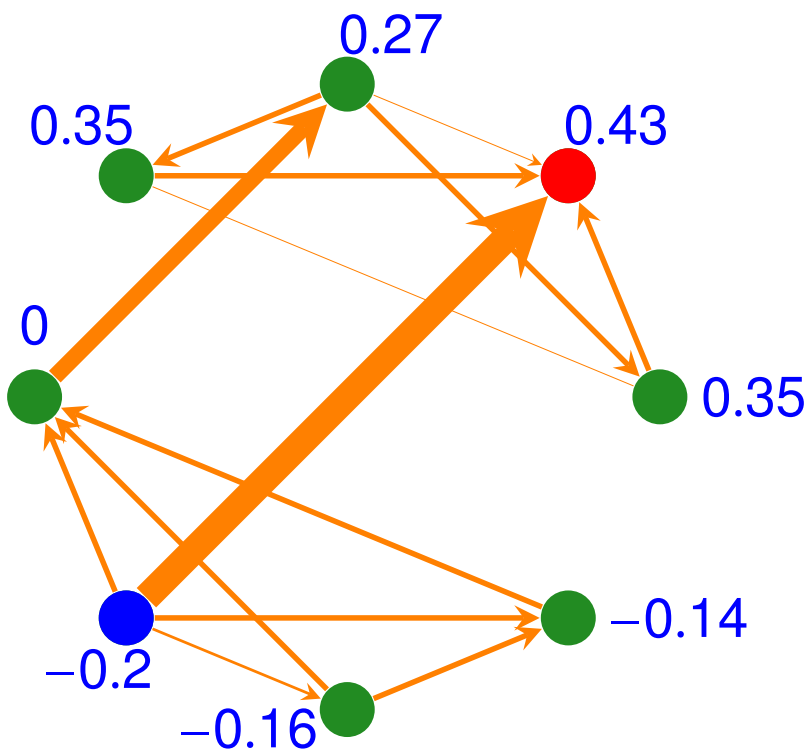
$\phi = L^+ \mathbf{b}_{uv}$=vertex potentials

$f_{xy} = \phi_y - \phi_x = \mathbf{b}_{xy}^T L^+ \mathbf{b}_{uv}$

Effective resistance

$$R_{uv} = \mathbf{b}_{uv}^T L^+ \mathbf{b}_{uv}$$

Note: defined for any pair (u, v).



Inject current at u, take out at v.

$\phi = L^+ \mathbf{b}_{uv}$=vertex potentials

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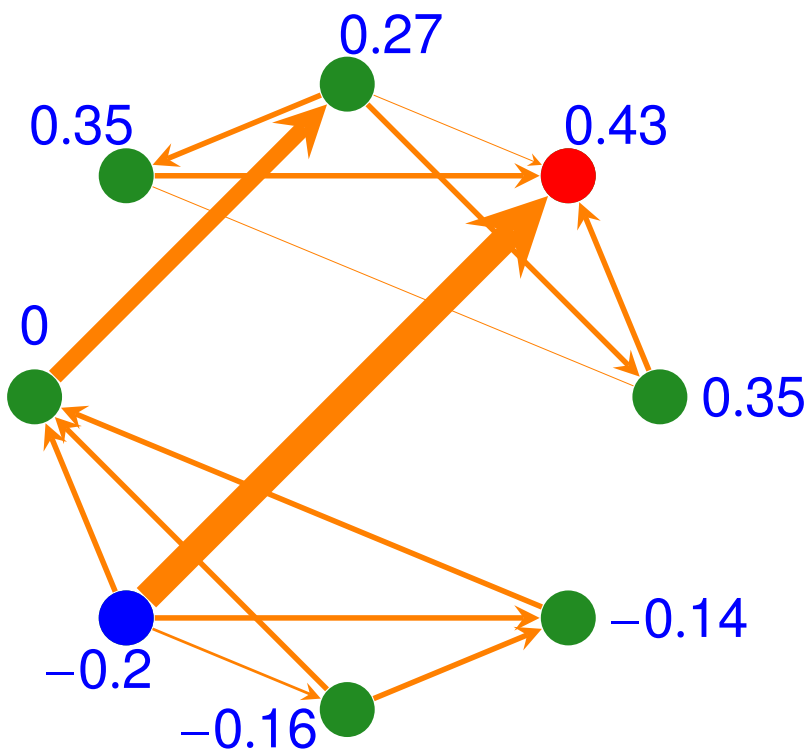

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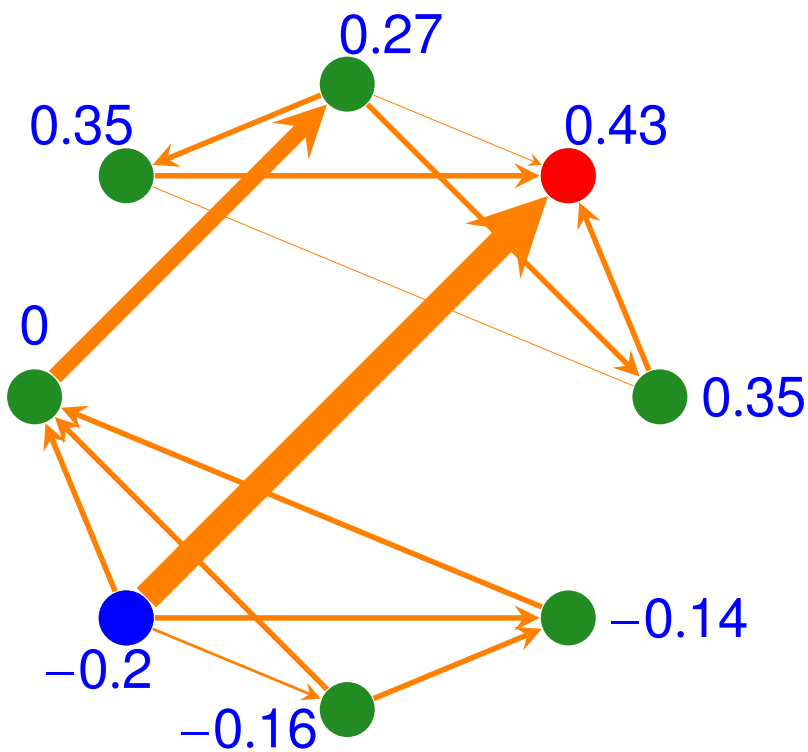

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R_{uv}= fraction of $||f||_2^2$ contributed by $e = (u, v)$

Given:

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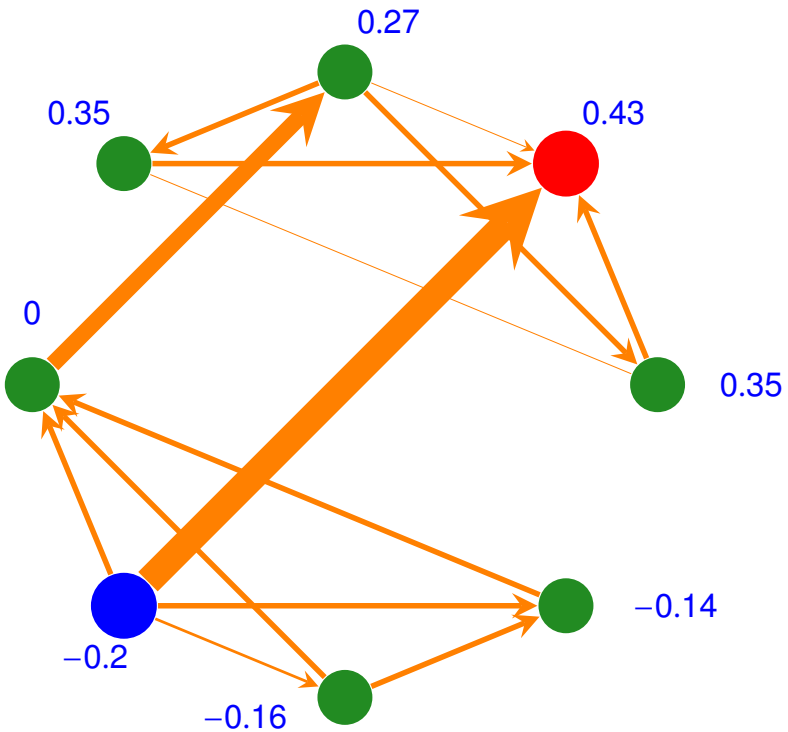

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Compute $f = \mathbf{B}\phi$=currents on edges

Check if

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0	0	0.01	0	**200**	1	0	2	0	0	0

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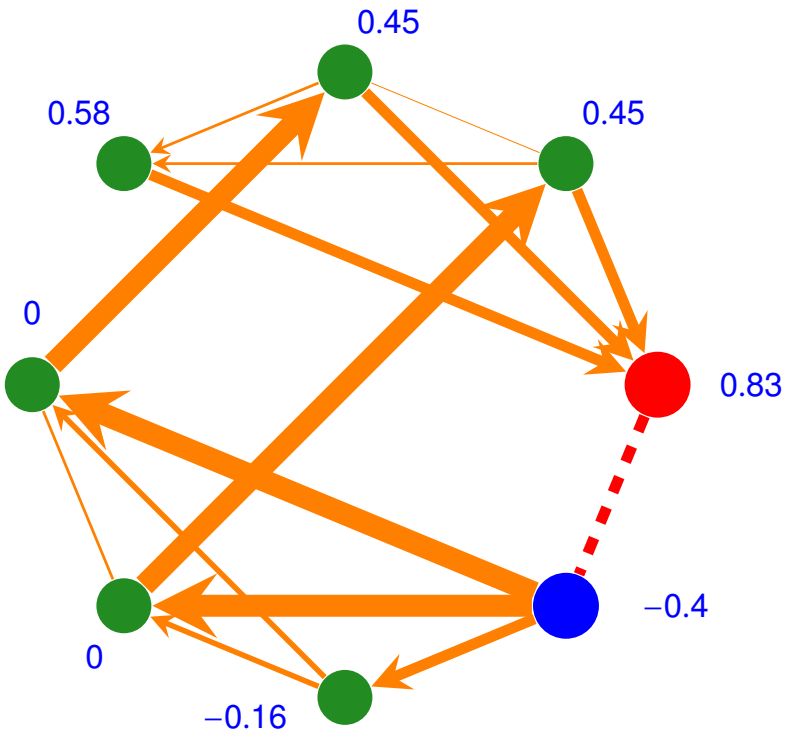

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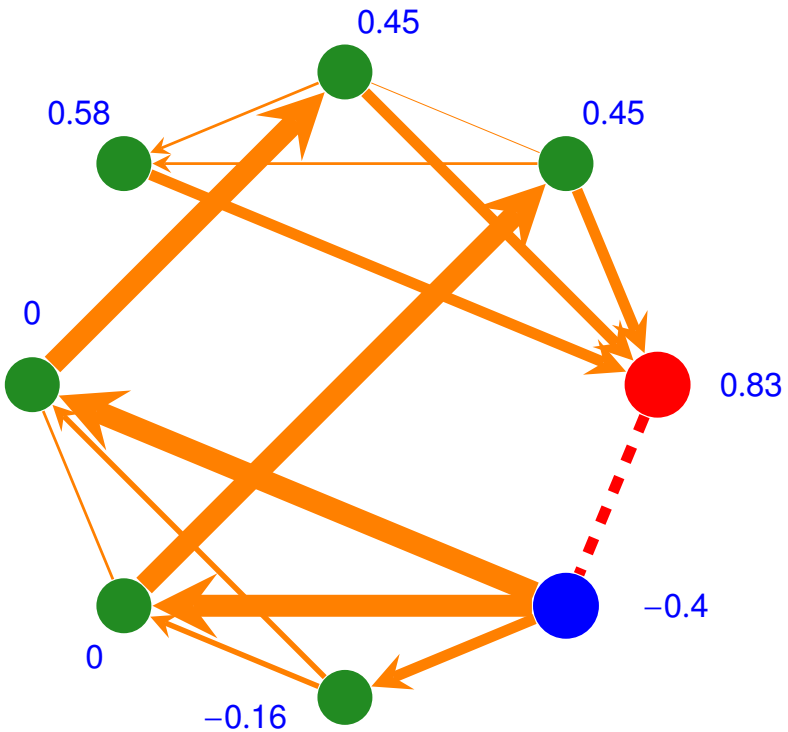

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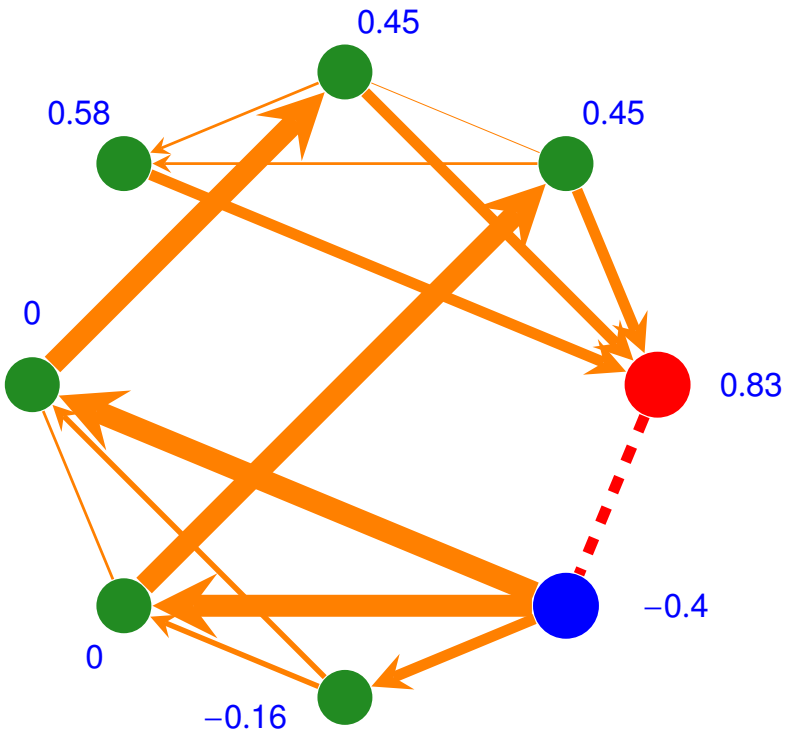

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Compute $\Delta = \mathbf{B}\phi \approx$ currents on edges

Check if

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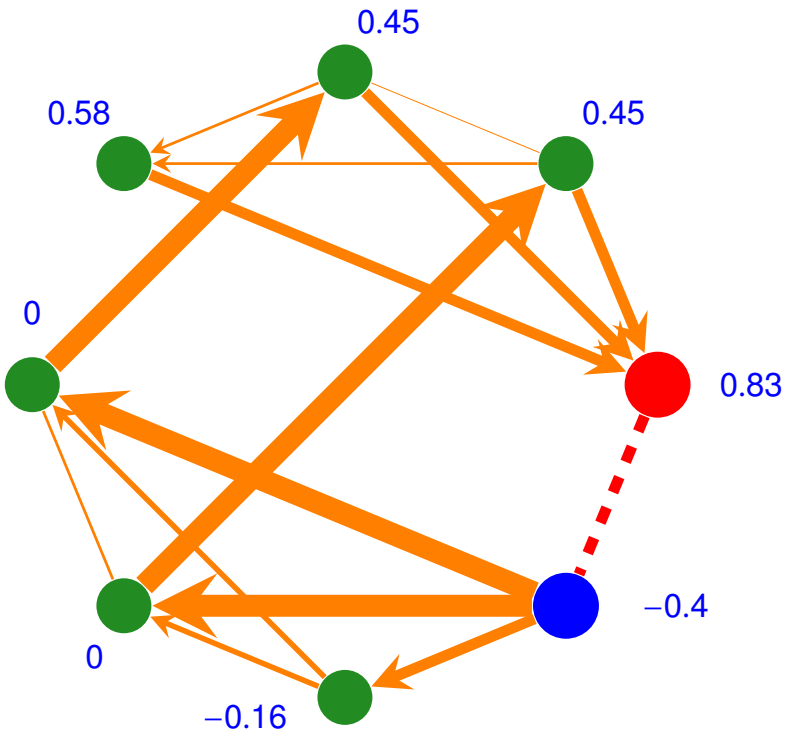

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Linear sketching and sparse recovery

0	0	0.01	0	**200**	1	0	2	0	0	0

Let y be a vector of reals. Then $i \in [n]$ is an ℓ_2-heavy hitter if

$$y_i^2 \geq \eta ||y||_2^2.$$

Lemma (ℓ_2-heavy hitters; CountSketch)

For any $\eta > 0$ there exists a (randomized) sketch in dimension $\frac{1}{\eta} \log^{O(1)} n$ from which one reconstruct all η-heavy hitters. The recovery works in time $\frac{1}{\eta} \cdot \log^{O(1)} n$.

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Sketching the edge incidence matrix

$$S \cdot \mathbf{B} = S \cdot \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \end{bmatrix}$$

Apply ℓ_2-heavy hitters sketch S to every column $\mathbf{b}_u, u \in V$ of $\mathbf{B}$

Store the n sketches, $n \cdot \log^{O(1)} n$ space.

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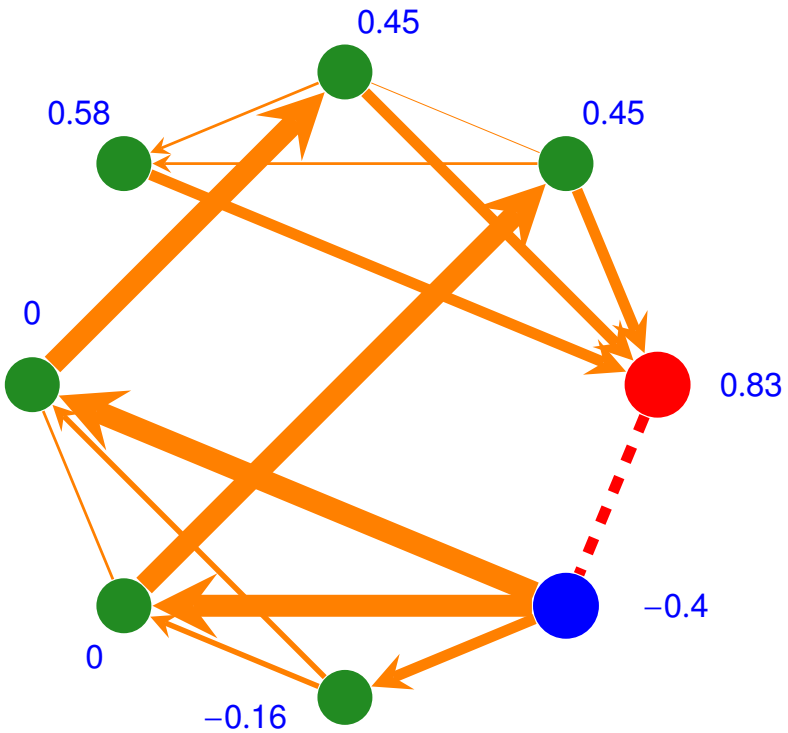

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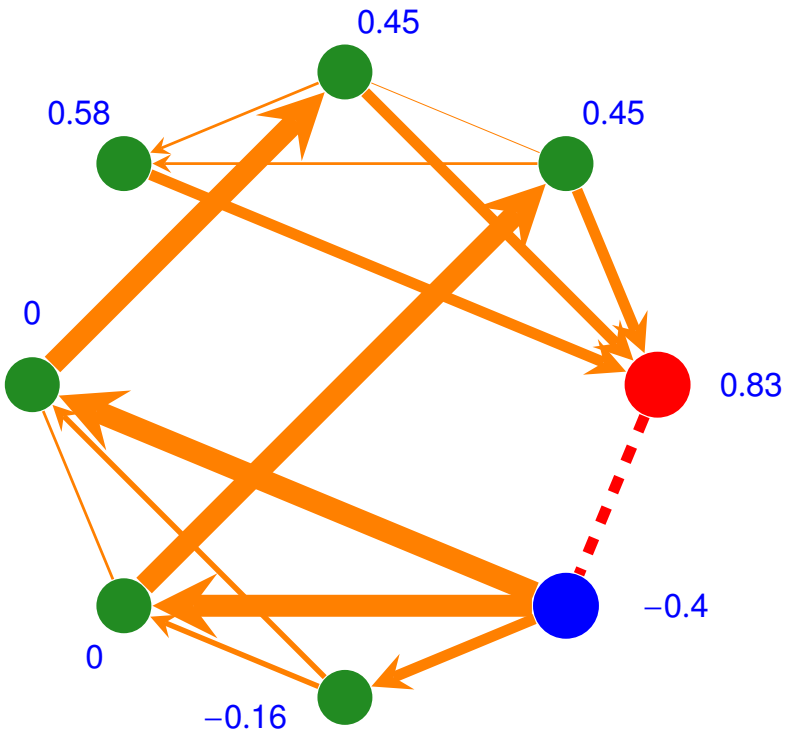

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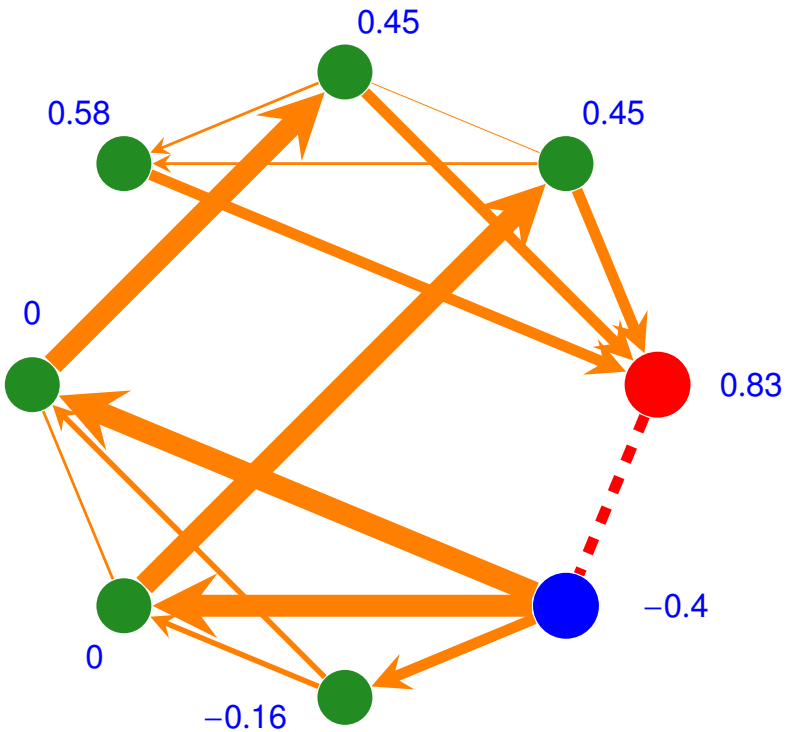

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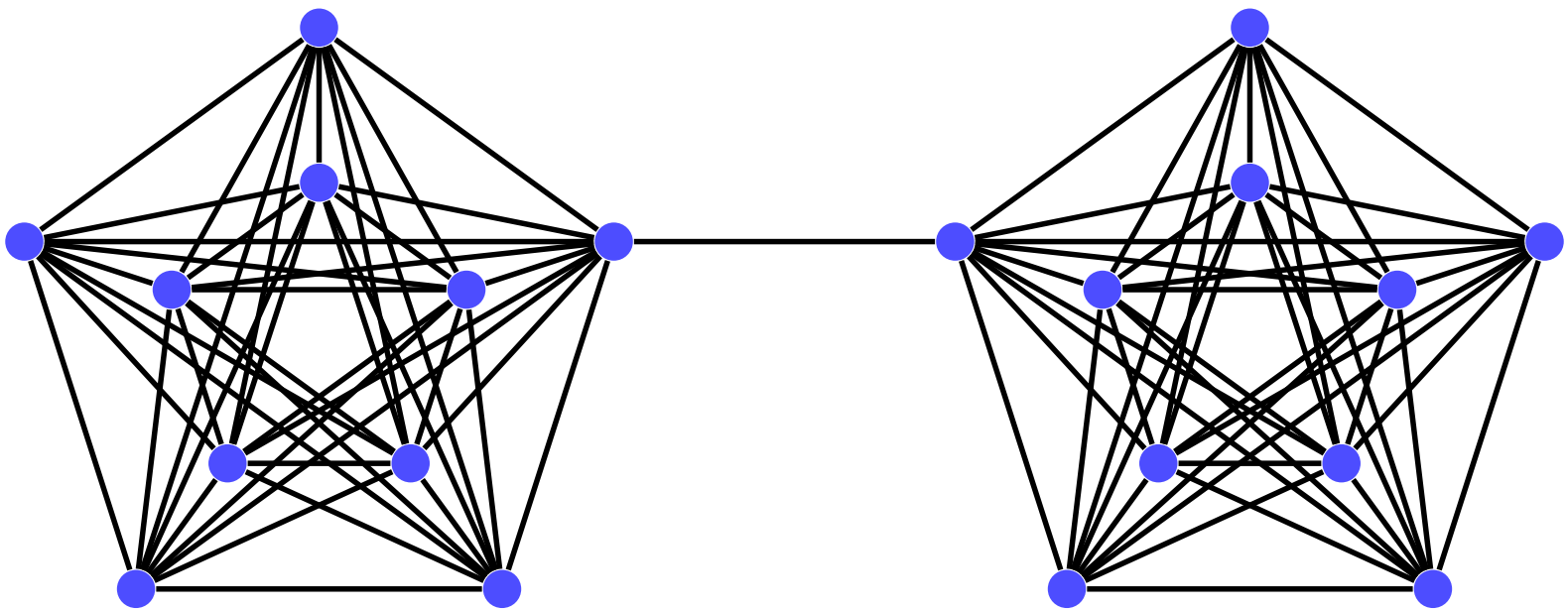
using heavy-hitters sketch S

0	0	0.01	0	200	1	0	2	0	**0**	0

Testing a pair of vertices – summary

Connect a battery to a pair of vertices $u, v \in V$

Edge e is high effective resistance iff the heat created by this edge in this setting is a non-trivial fraction of the total heat.



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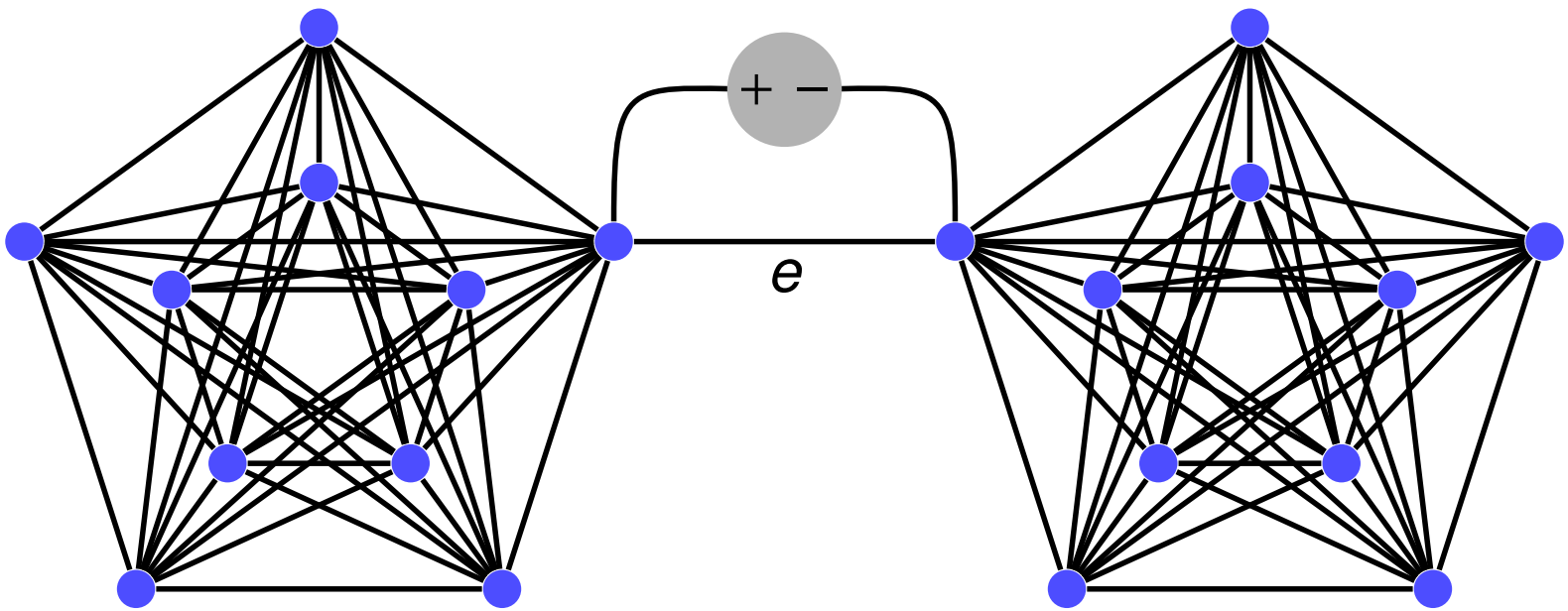

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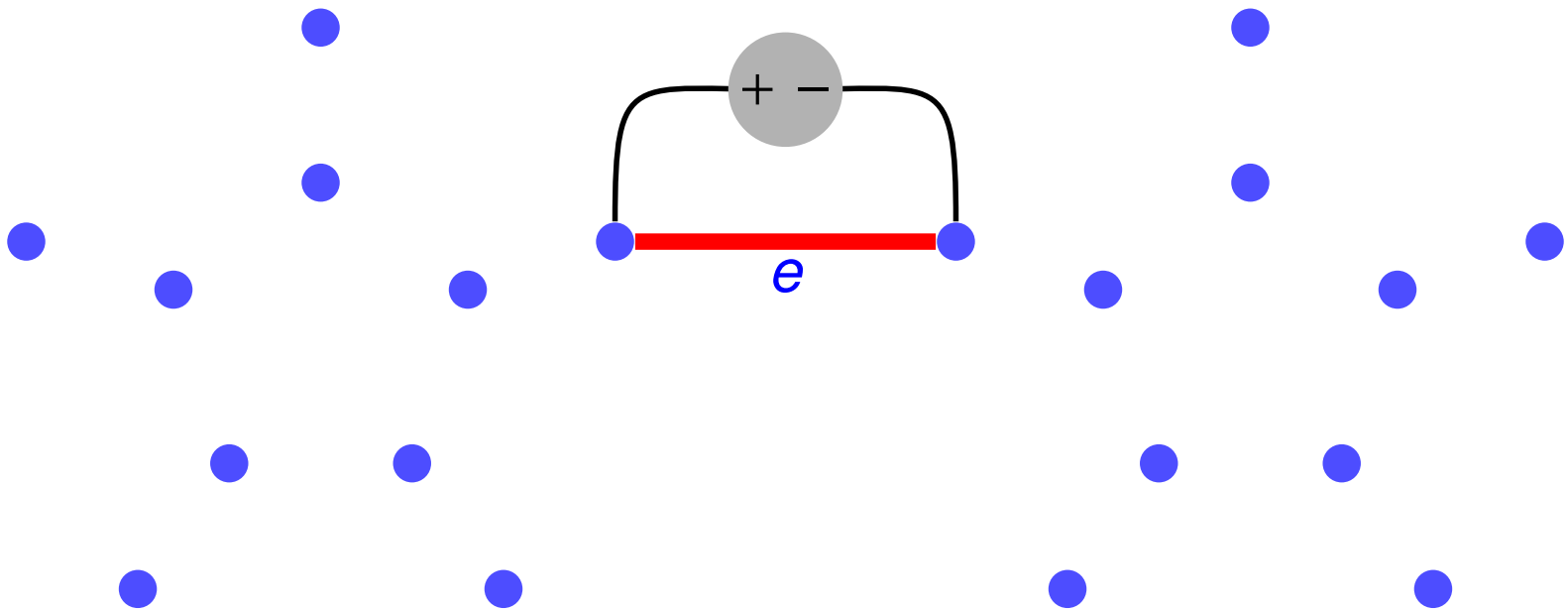

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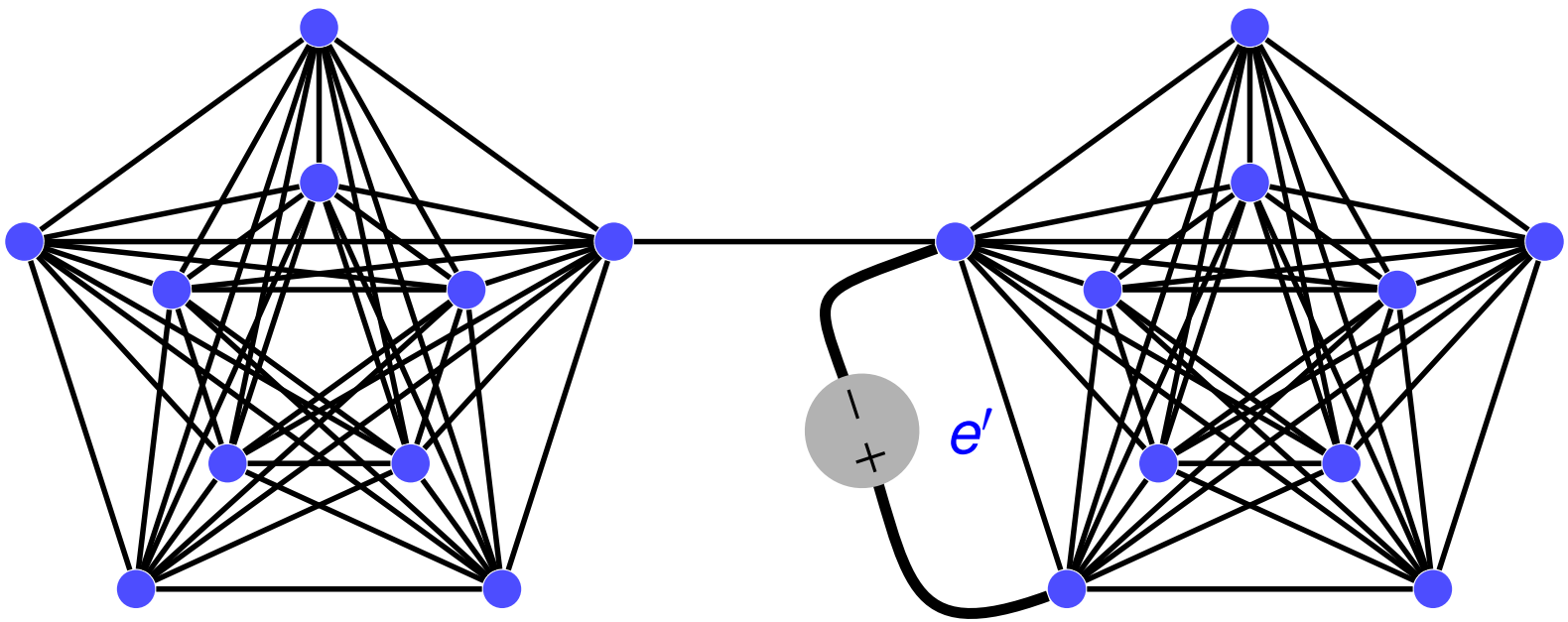

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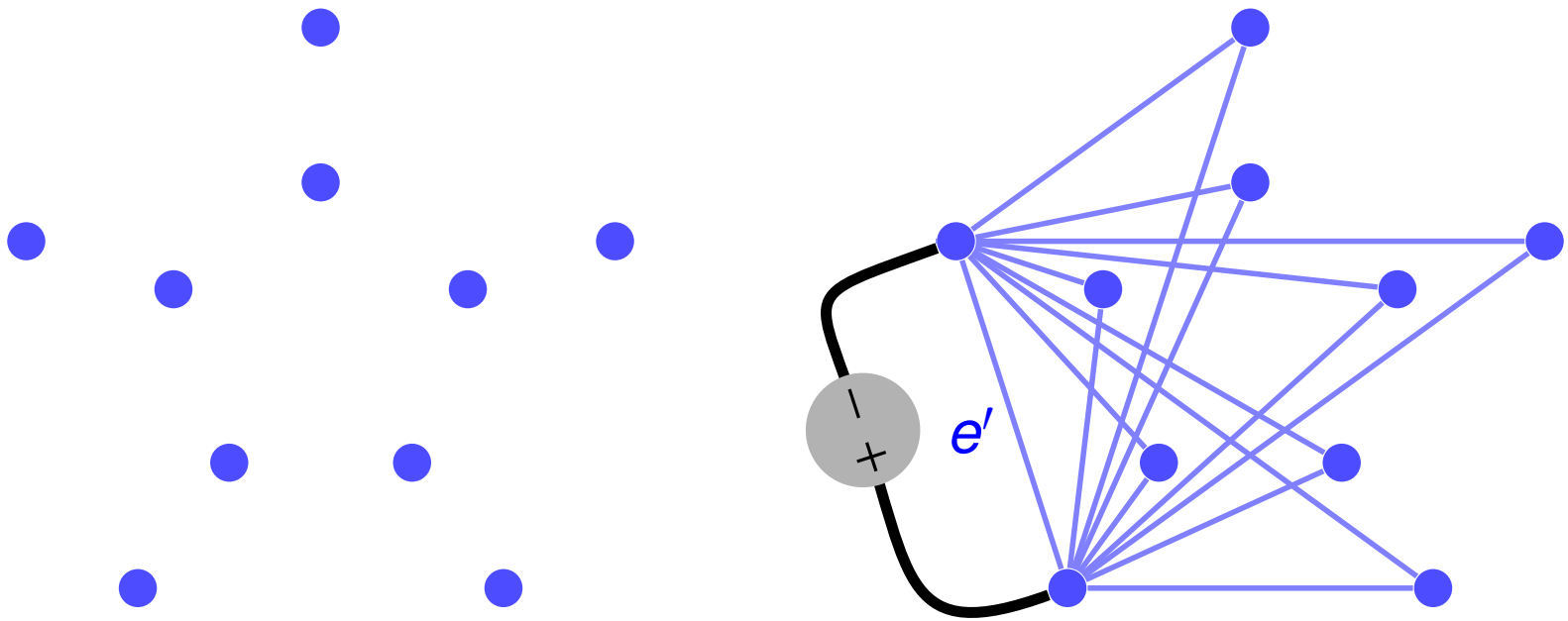

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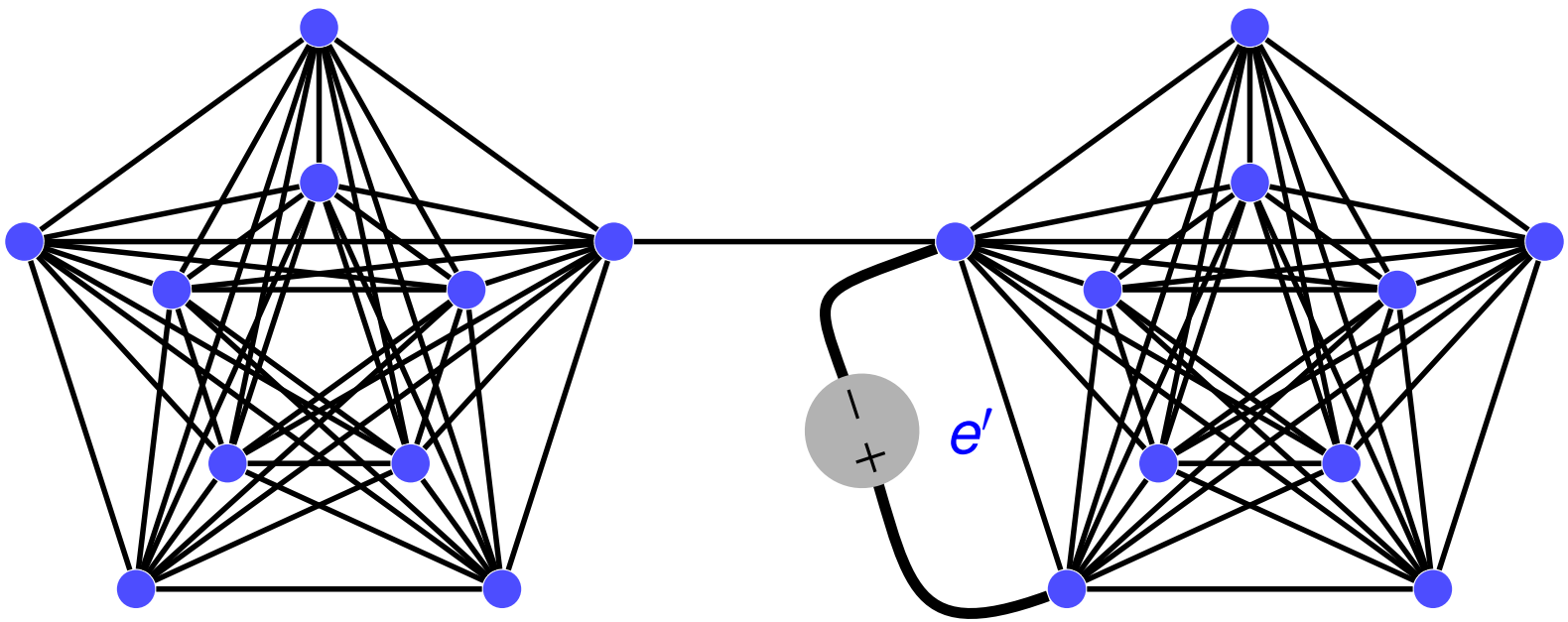


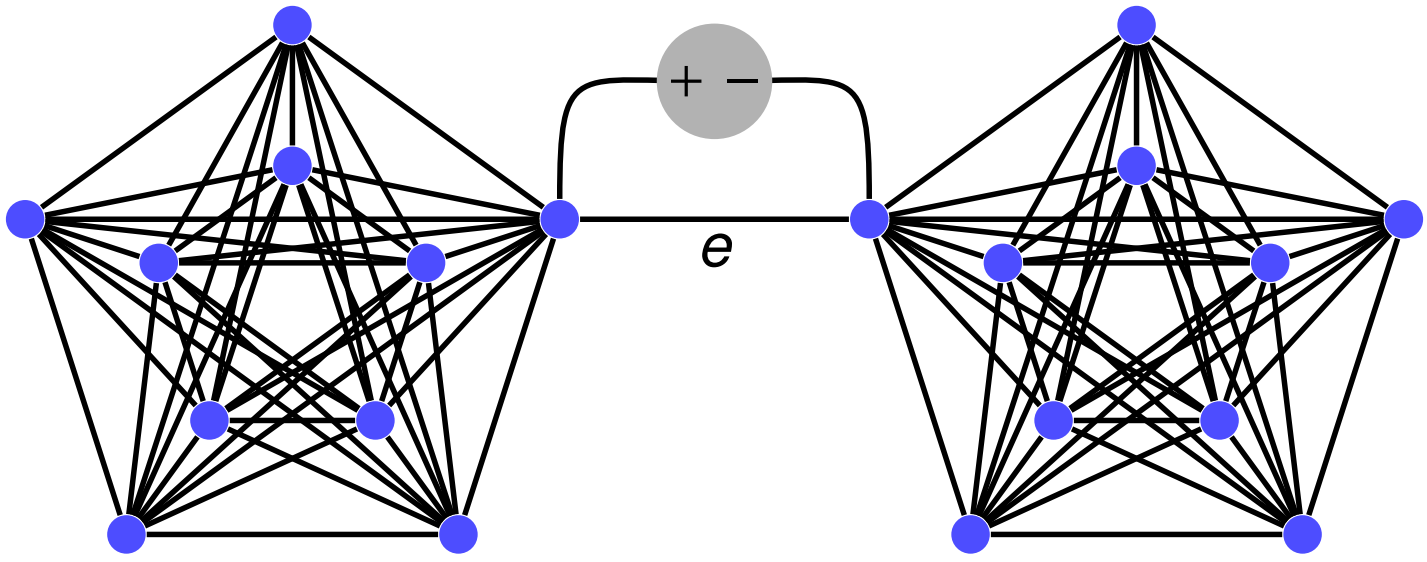
Find all heavy edges with $O(n)$ experiments?

Find shortcuts in wiring

Find shortcuts in wiring

Find wires with high effective resistance

Edges with high effective resistance



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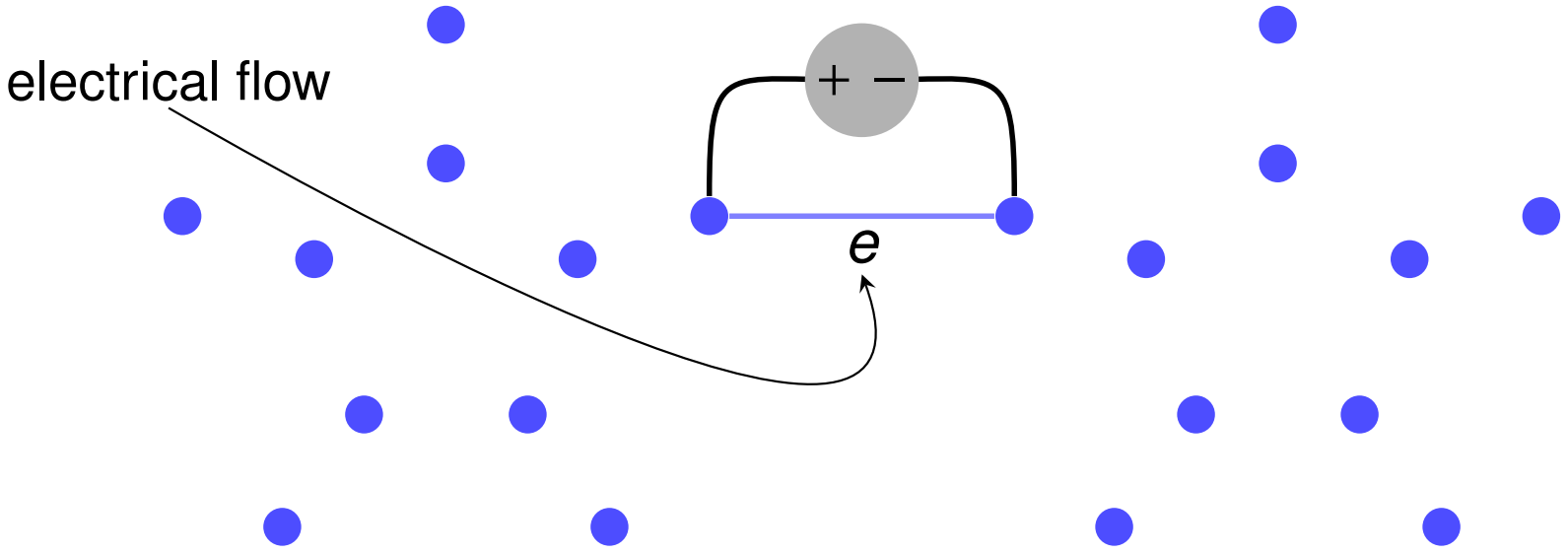

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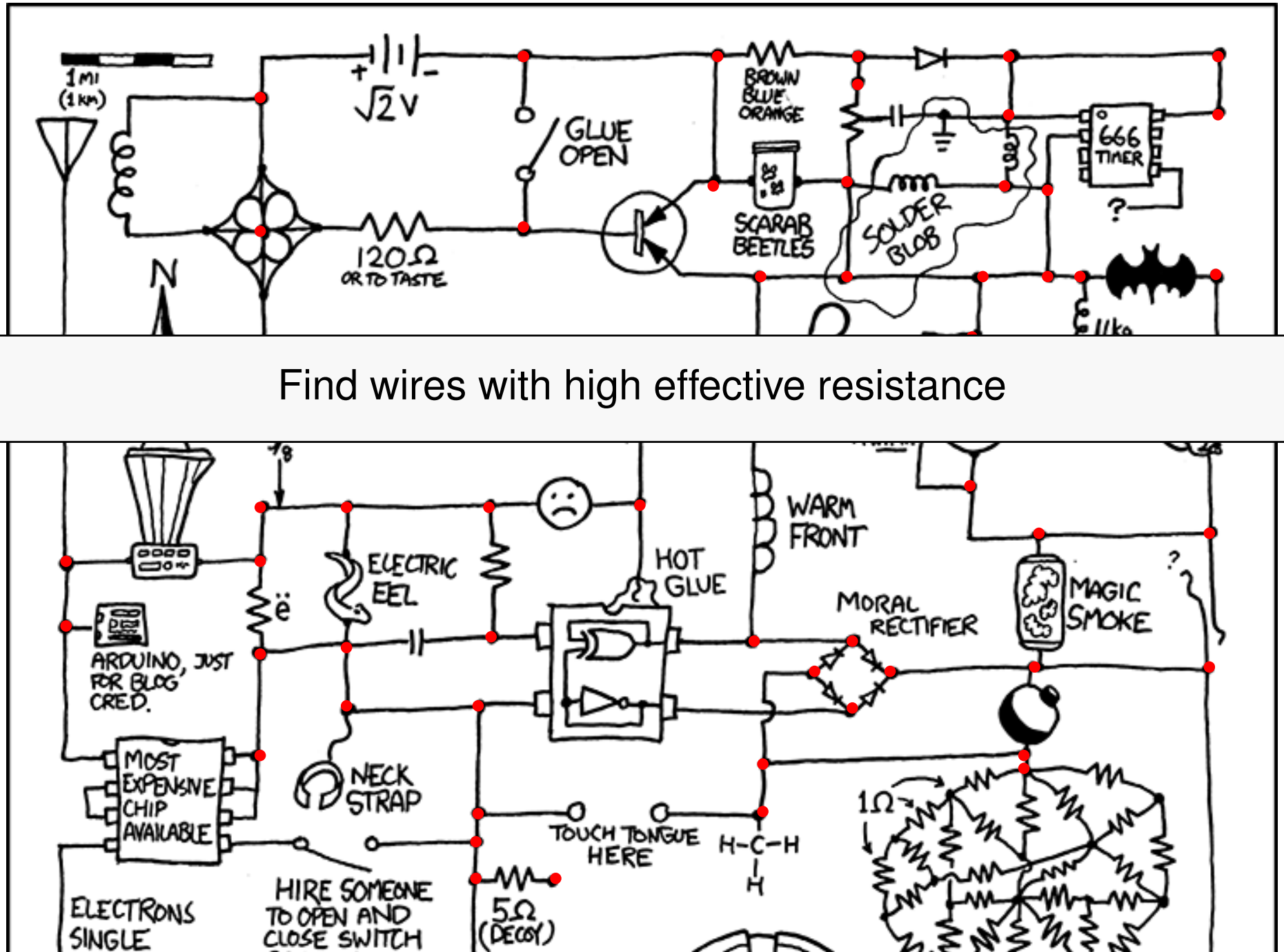

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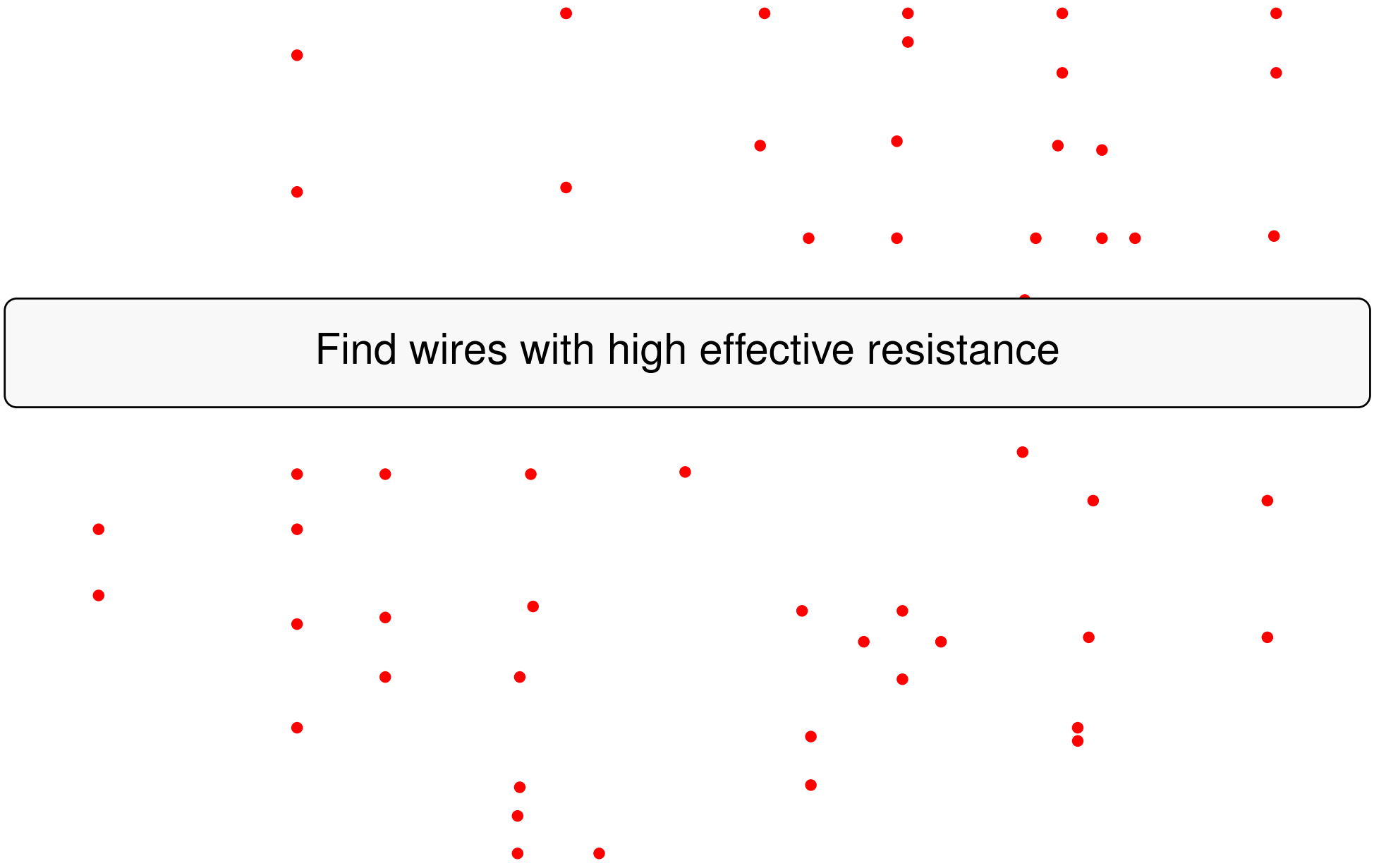

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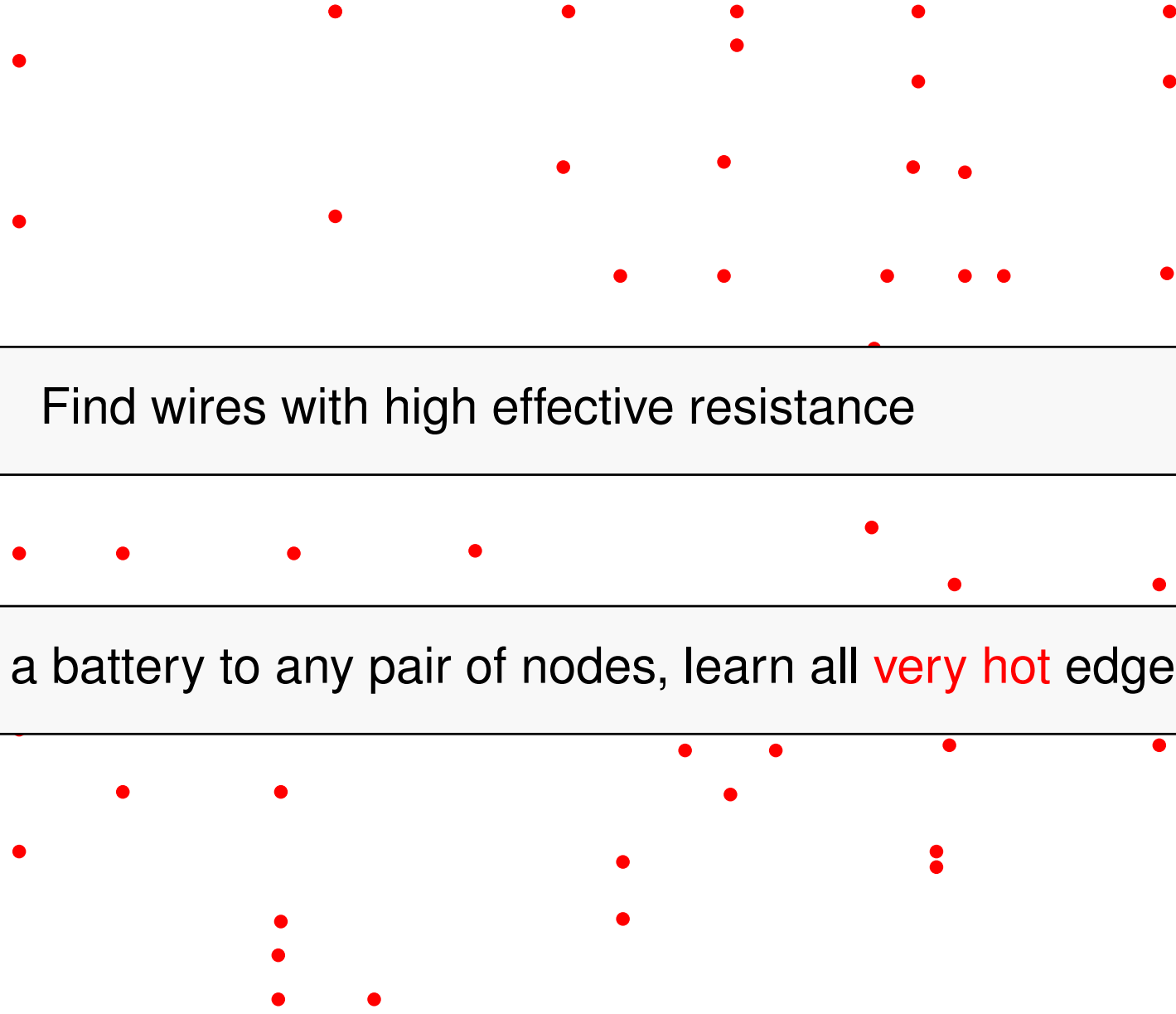

Find wires with high effective resistance

Can attach a battery to any pair of nodes, learn all very hot edges

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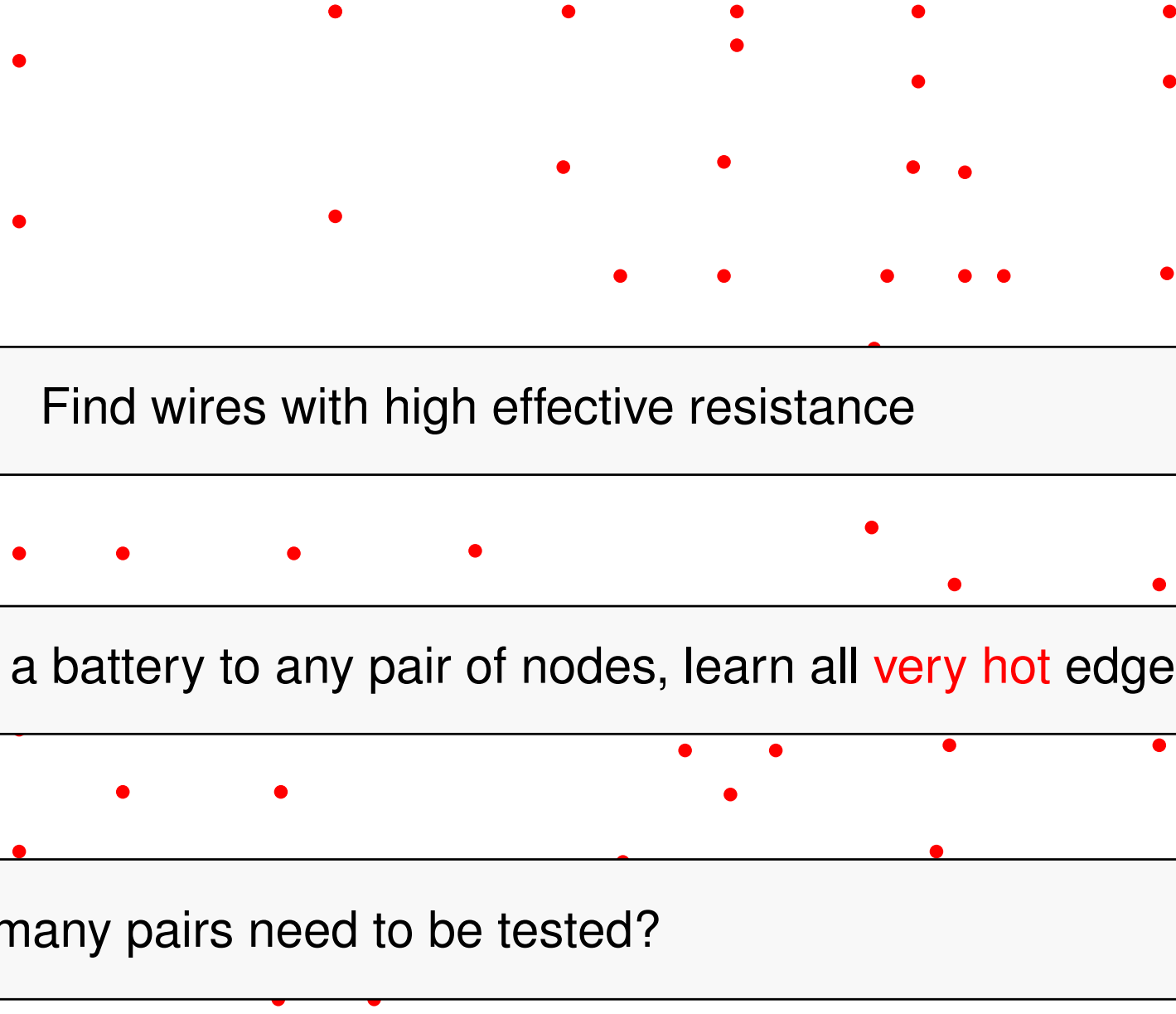

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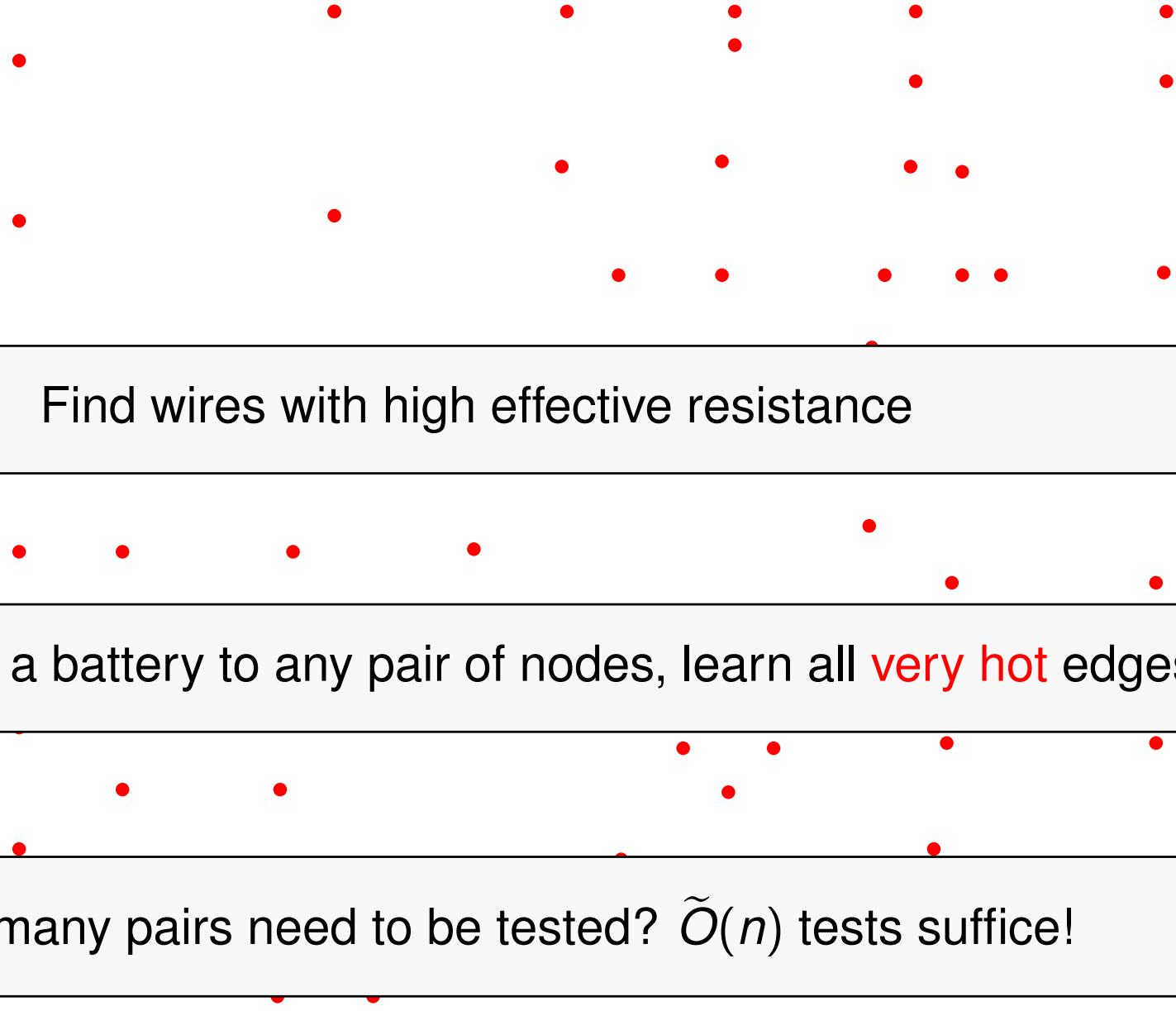
How many pairs need to be tested?

Find shortcuts in wiring



Find wires with high effective resistance

Can attach a battery to any pair of nodes, learn all very hot edges

How many pairs need to be tested? $\tilde{O}(n)$ tests suffice!

1. Spectral sparsification
2. Dynamic streaming and linear sketching
3. Spectral sparsification using small sketches
4. Fast decoding

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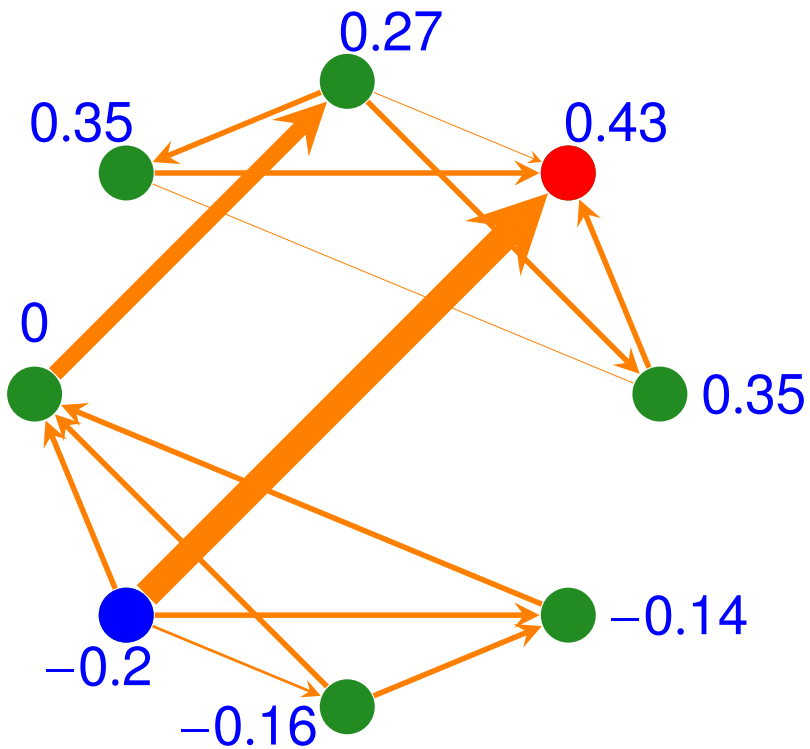
Effective resistance



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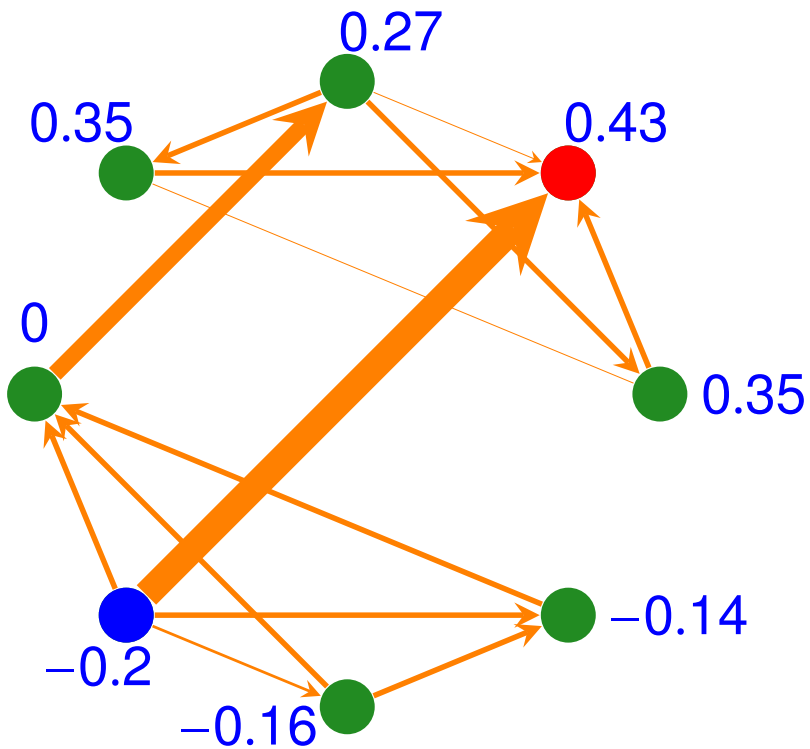

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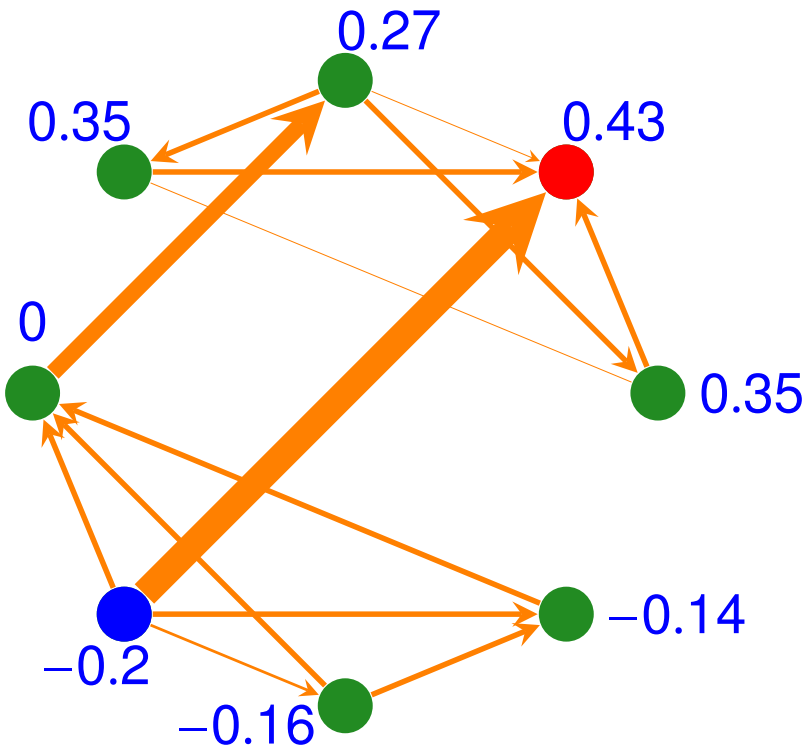
Find pair $a, b \in V$ such that (u, v) carries a lot of $a \to b$ flow?

Effective resistance



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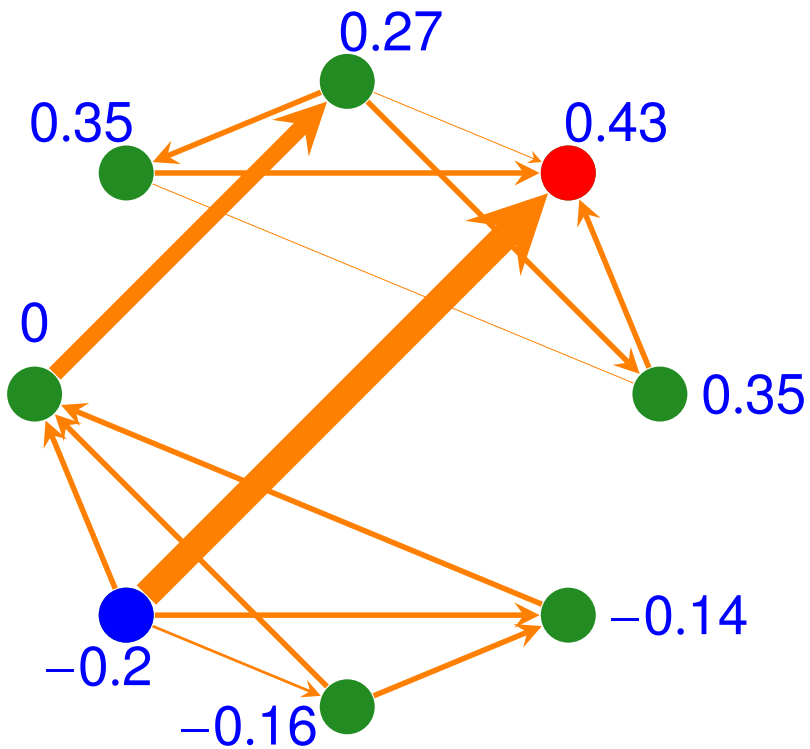
$$\frac{(\tau_u - \tau_v)^2}{R_{ab}} \gtrsim 1/\log n$$

where $\tau = L^+ \mathbf{b}_{ab}$.

Effective resistance



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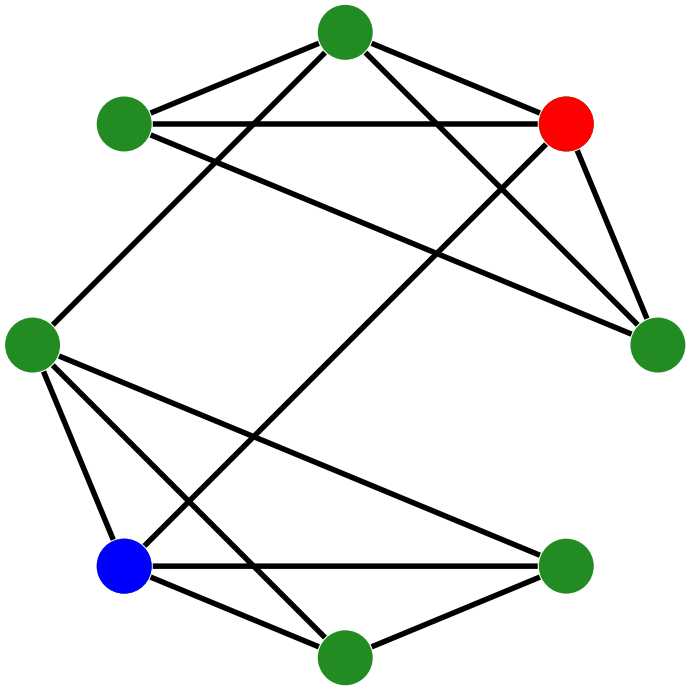


Inject current at u, take out at v.

$\phi = L^+ \mathbf{b}_{uv}$=vertex potentials

$f = \mathbf{B}\phi$=currents on edges

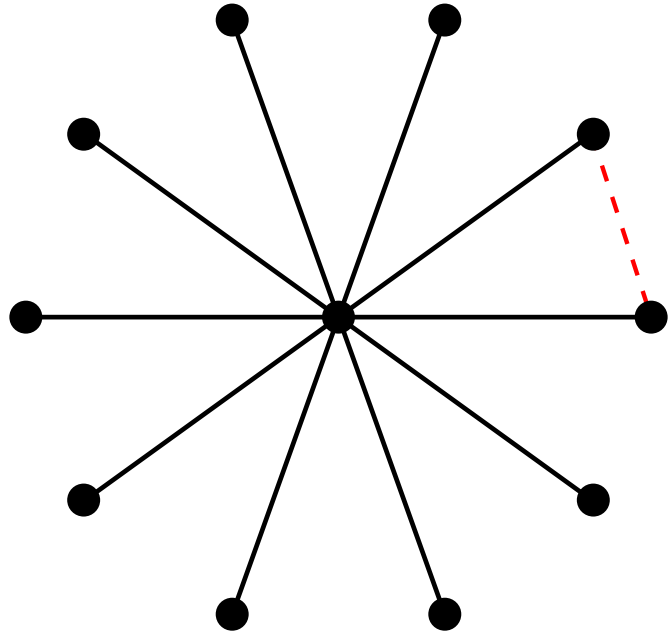
We have

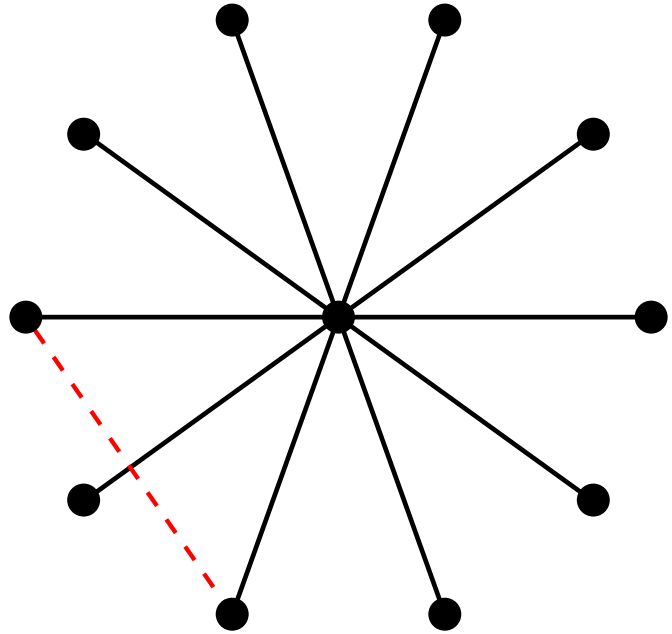
$$R_{uv} = \frac{f_{uv}^2}{||f||^2} = \frac{(\phi_u - \phi_v)^2}{R_{uv}}.$$

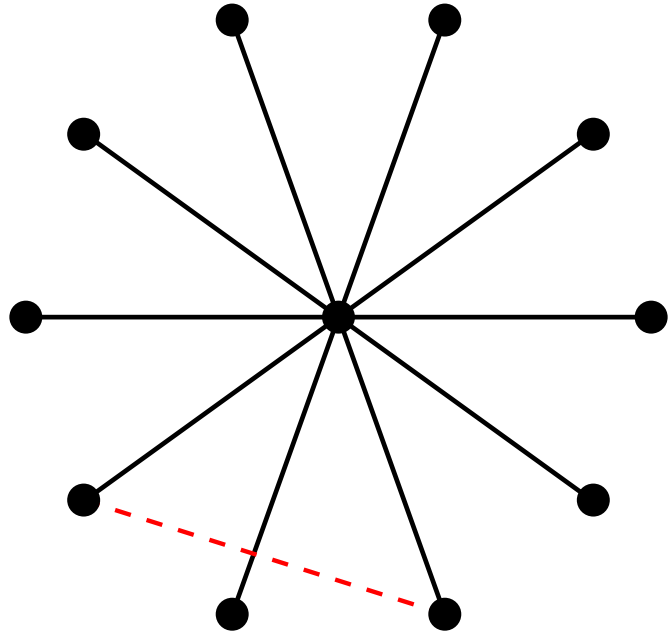
Find pair $a, b \in V$ such that (u, v) carries a lot of $a \to b$ flow?

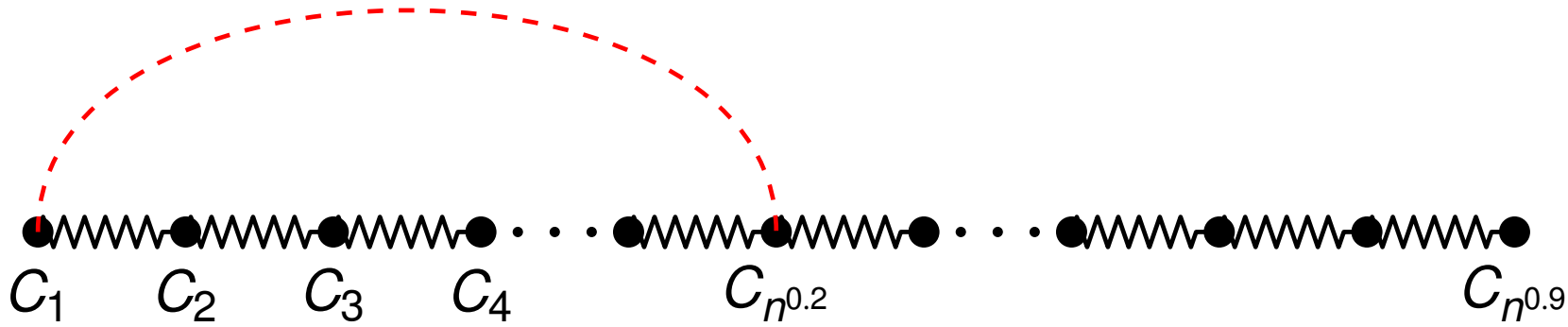
$$\frac{(\tau_u - \tau_v)^2}{R_{ab}} \gtrsim \frac{(\phi_u - \phi_v)^2}{R_{uv}}$$

where $\tau = L^+ \mathbf{b}_{ab}$.

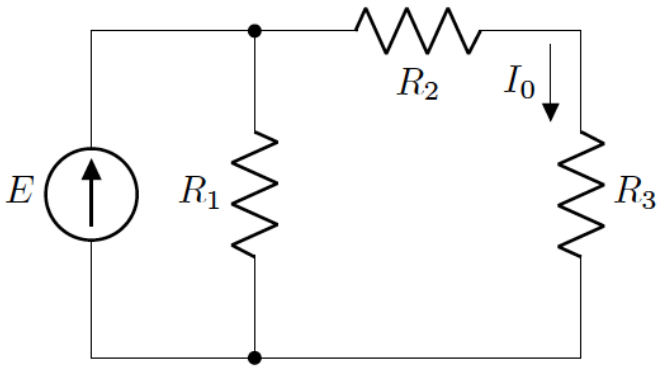




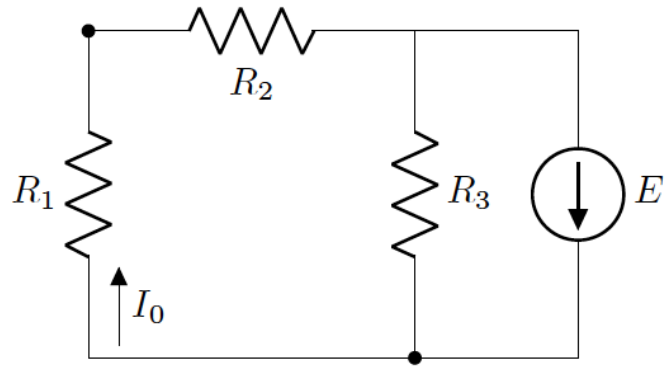



C_1
C_2
C_3
C_4
$C_{n^{0.2}}$
$C_{n^{0.9}}$

Reciprocity theorem



(a)



(b)

$u \to v$ flow results in potentials $\phi = L^{+}\mathbf{b}_{uv}$

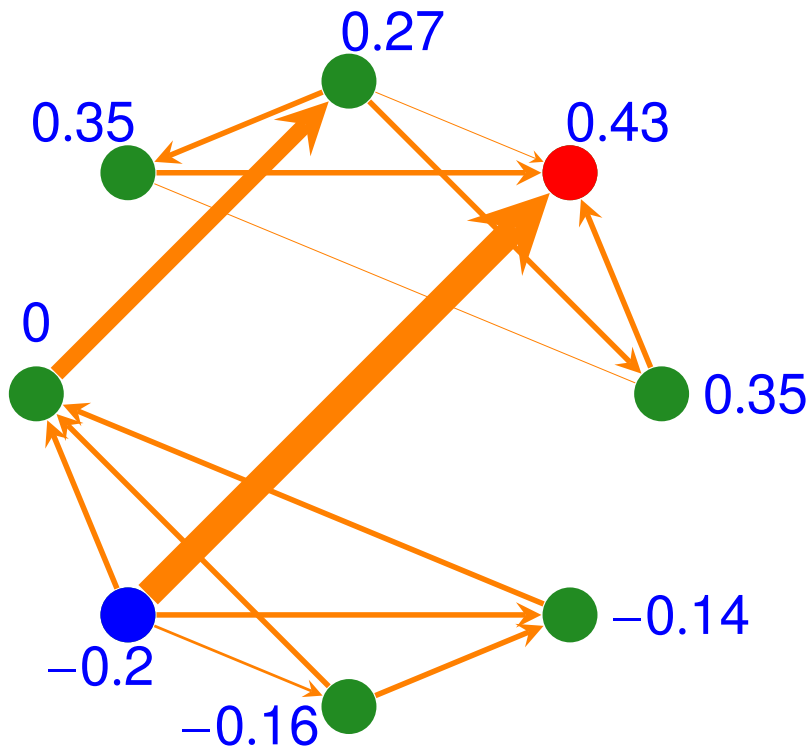
$a \to b$ flow results in potentials $\tau = L^{+}\mathbf{b}_{ab}$

$$\tau_v - \tau_u = \mathbf{b}_{uv}^T \tau = \mathbf{b}_{uv}^T L^{+} \mathbf{b}_{ab}$$

and

$$\phi_b - \phi_a = \mathbf{b}_{ab}^T \phi = \mathbf{b}_{ab}^T L^{+} \mathbf{b}_{uv}$$

Effective resistance

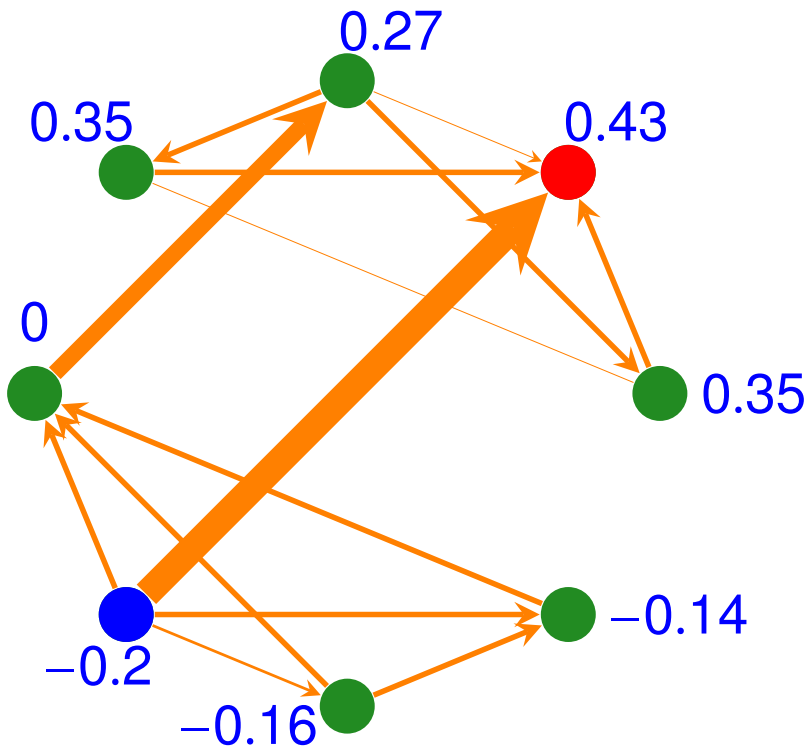


Find pair $a, b \in V$ such that (u, v) carries a lot of $a \to b$ flow?

$$\frac{(\tau_u - \tau_v)^2}{R_{ab}} \approx \frac{(\phi_u - \phi_v)^2}{R_{uv}}$$

where $\tau = L^+ \mathbf{b}_{ab}$.

Potential embedding



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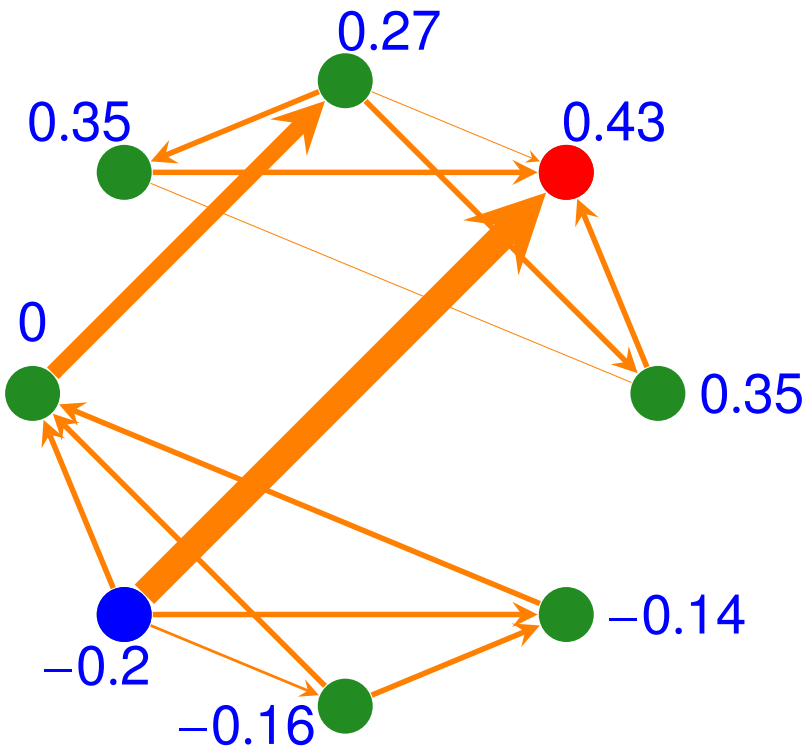
Potential embedding of (u, v):

Potential embedding



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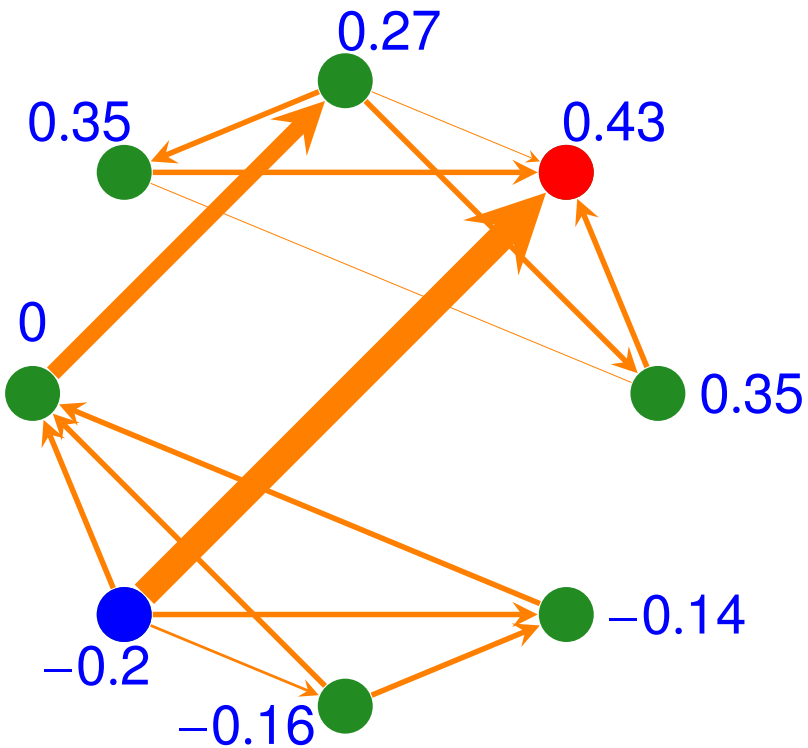
Potential embedding of (u, v):

For $x \in V$

$$\phi_v - \phi_u = (\phi_v - \phi_x) + (\phi_x - \phi_u)$$

Potential embedding



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Potential embedding of (u, v):

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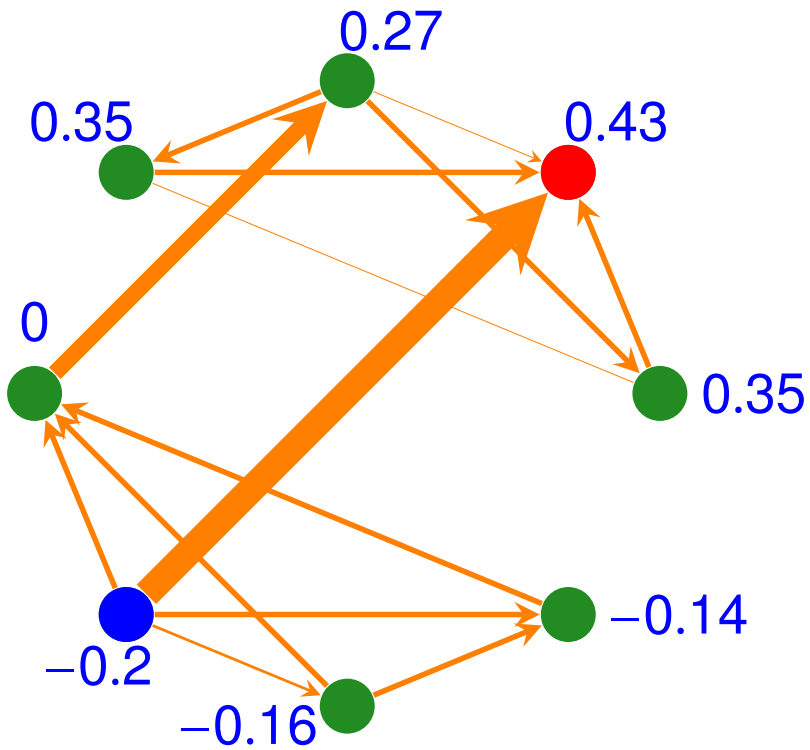
$$\phi_v - \phi_u = (\phi_v - \phi_x) + (\phi_x - \phi_u) = \mathbf{b}_{xv} L^+ \mathbf{b}_{uv} + \mathbf{b}_{ux} L^+ \mathbf{b}_{uv}$$

Potential embedding



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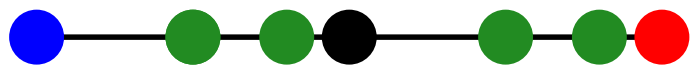

For $x \in V$

$$\phi_v - \phi_u = (\phi_v - \phi_x) + (\phi_x - \phi_u) = \mathbf{b}_{uv}^T L^+ \mathbf{b}_{ux} + \mathbf{b}_{uv}^T L^+ \mathbf{b}_{xv}$$

Potential embedding

Potential embedding of (u, v):

Let $\tau = L^{+}\mathbf{b}_{xu}$ denote $x \to u$ flow potentials:

$$(\tau_u - \tau_v)^2 = (\mathbf{b}_{ux} L^{+} \mathbf{b}_{uv})^2 \geq \frac{1}{4}(\phi_u - \phi_v)^2$$

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So (u, v) carries a lot of $x \to u$ flow?

Potential embedding

Potential embedding of (u, v):

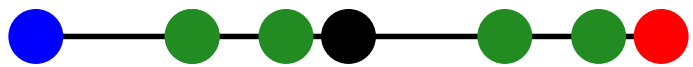
So

$$\frac{(\tau_u - \tau_v)^2}{R_{xu}} \gtrsim \frac{(\phi_u - \phi_v)^2}{R_{uv}} ?$$

Yes...

Potential embedding

Potential embedding of (u, v):



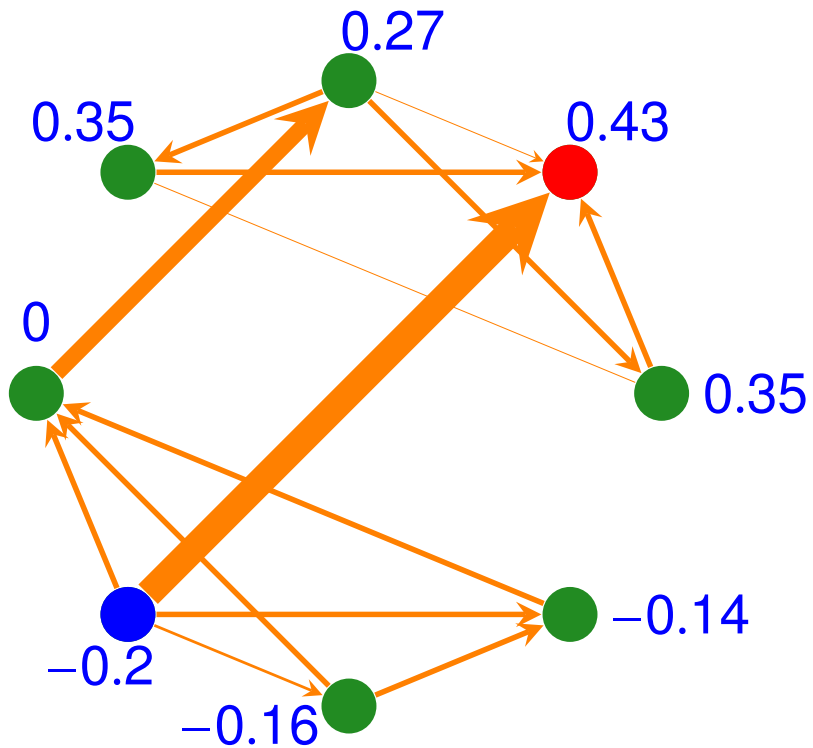
So

$$\frac{(\tau_u - \tau_v)^2}{R_{xu}} \gtrsim \frac{(\phi_u - \phi_v)^2}{R_{uv}}?$$

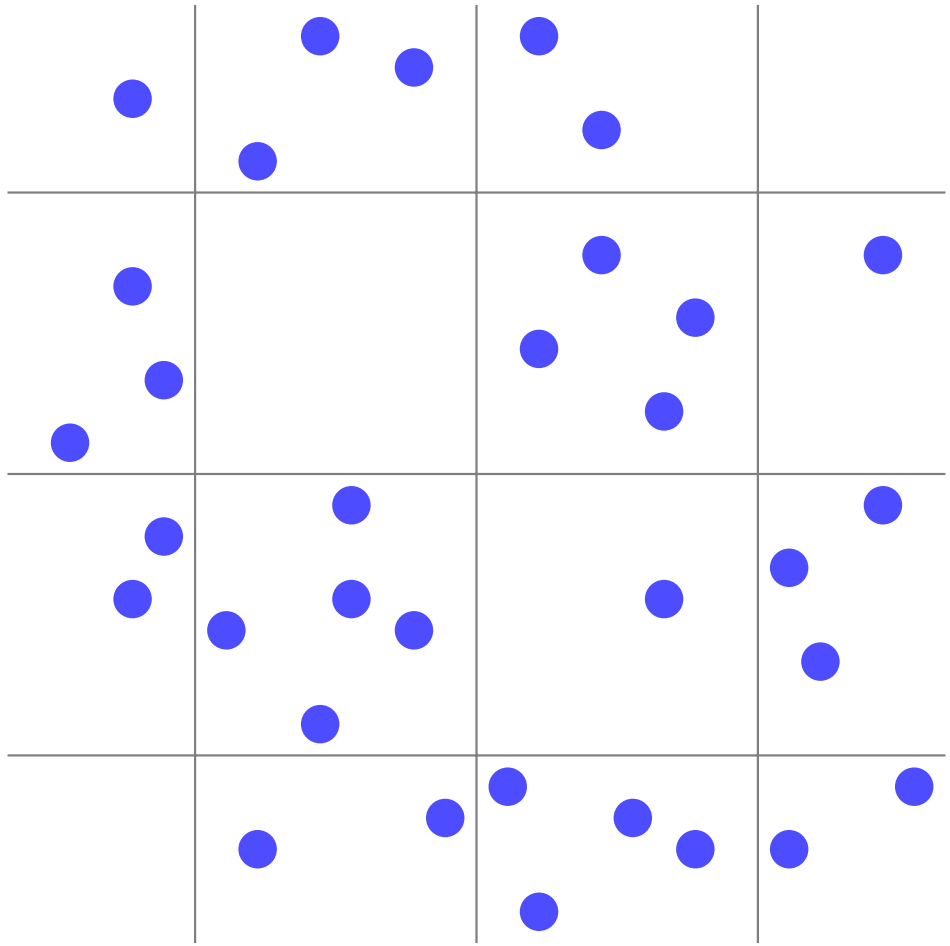
Yes... as long as $R_{xu} \lesssim R_{uv}$

To discover edge (u, v) with $R_{uv} \geq 1/\log n$:

1. Choose $x \in V$ with $R_{ux} \leq \log^{O(1)} n$
2. Send flow from x to u, list heavy edges
3. Send flow from x to v, list heavy edges

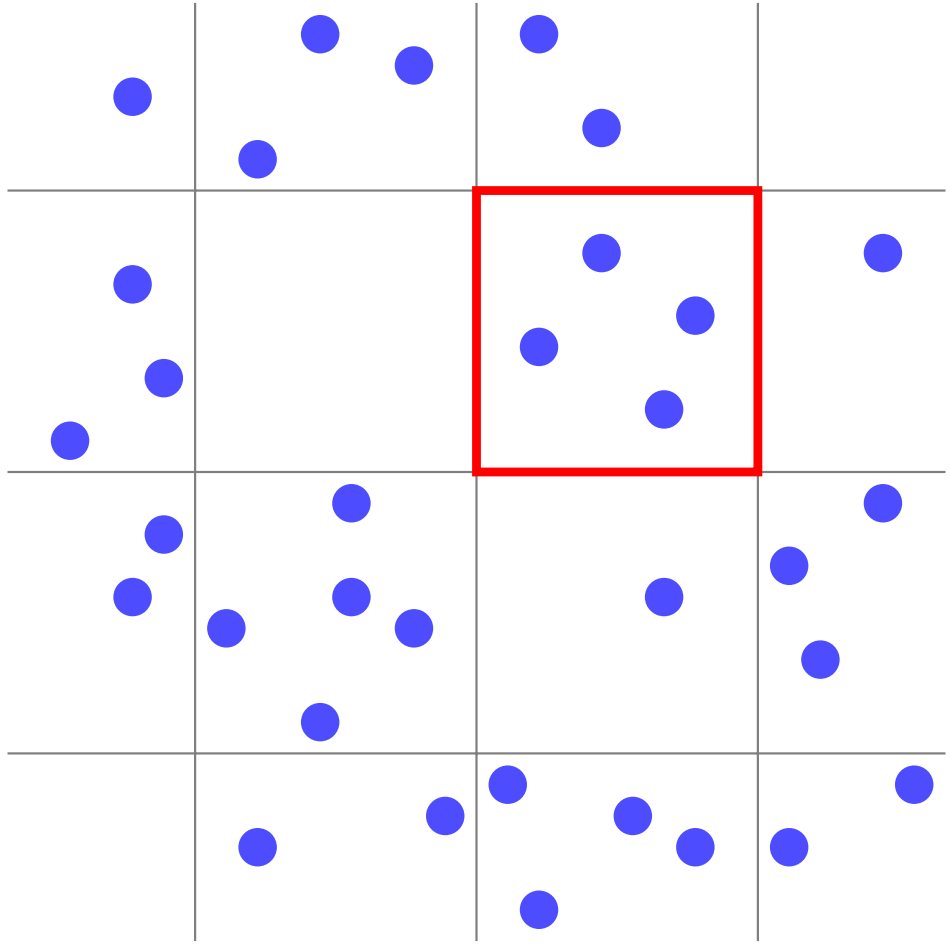


Our Approach



Partition vertices into clusters of bounded effective resistance diameter

Our Approach



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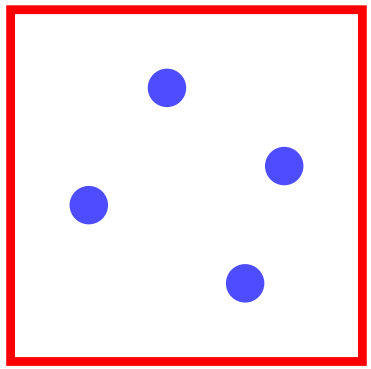
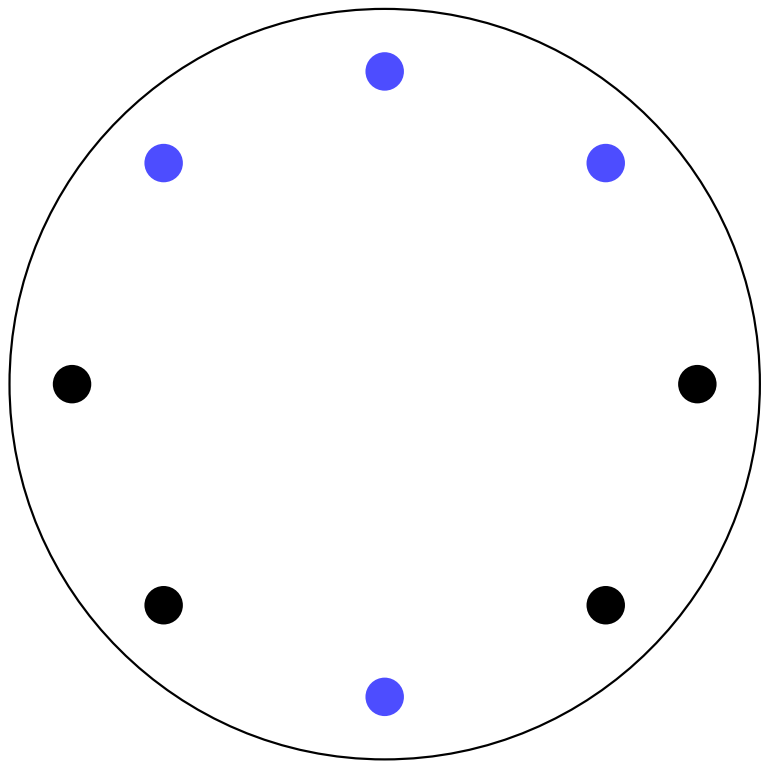


Pick representative vertex x in every cluster, try flows from x to all vertices in same cluster

This finds every heavy edge not cut by the partition



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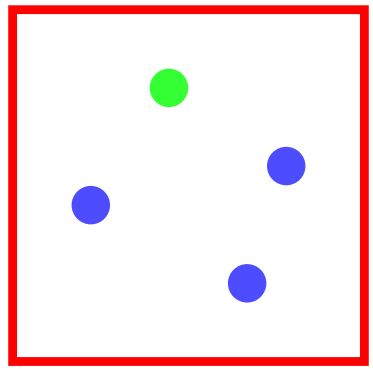

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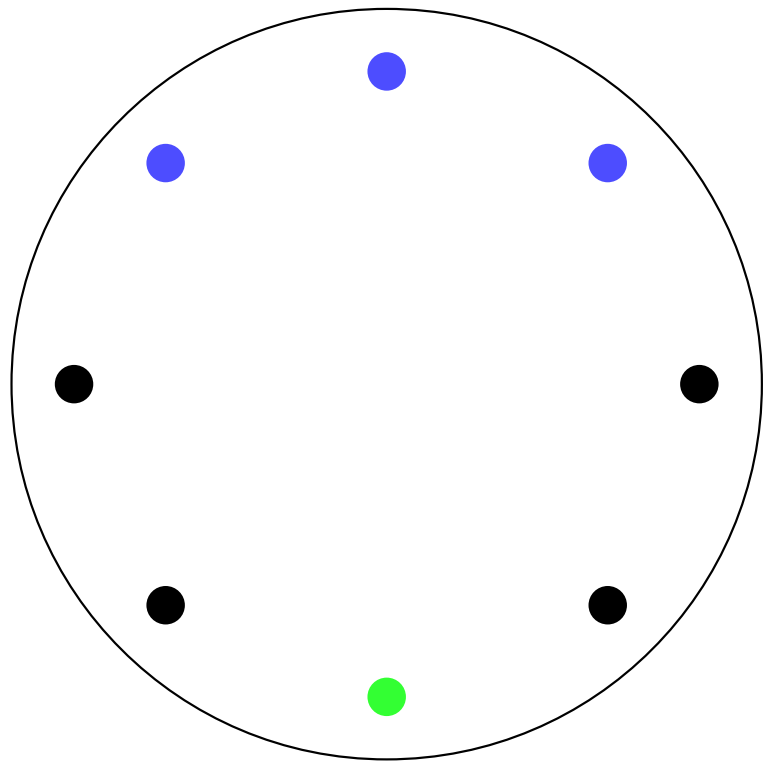

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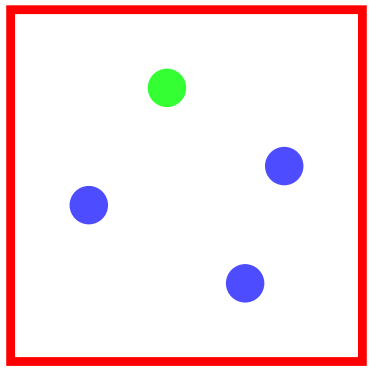

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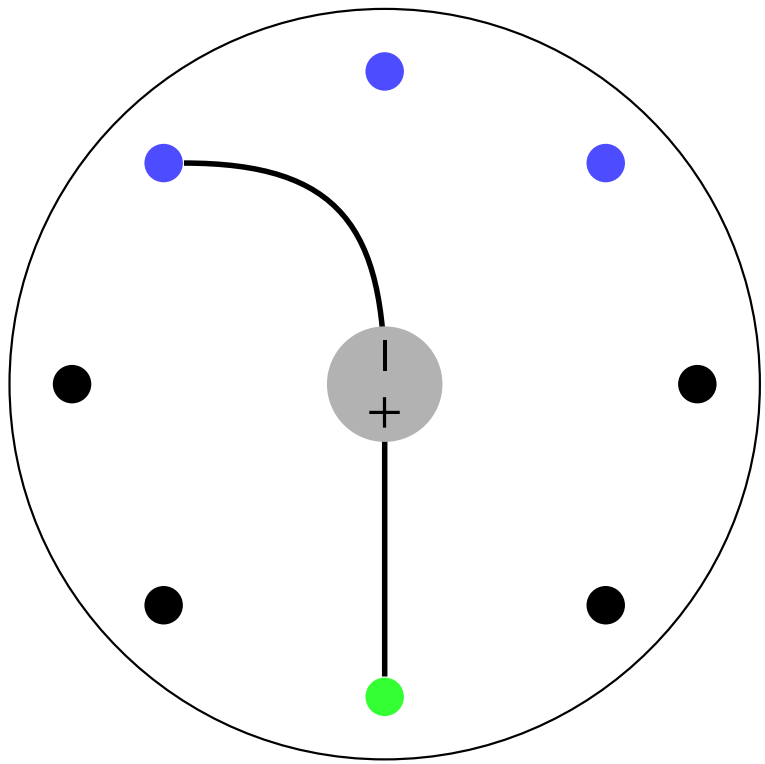

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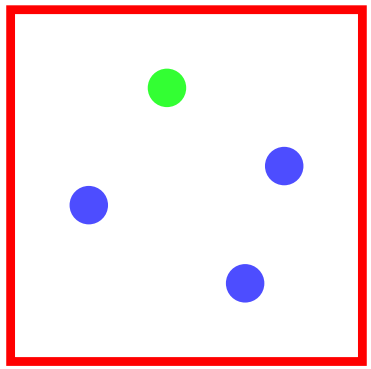

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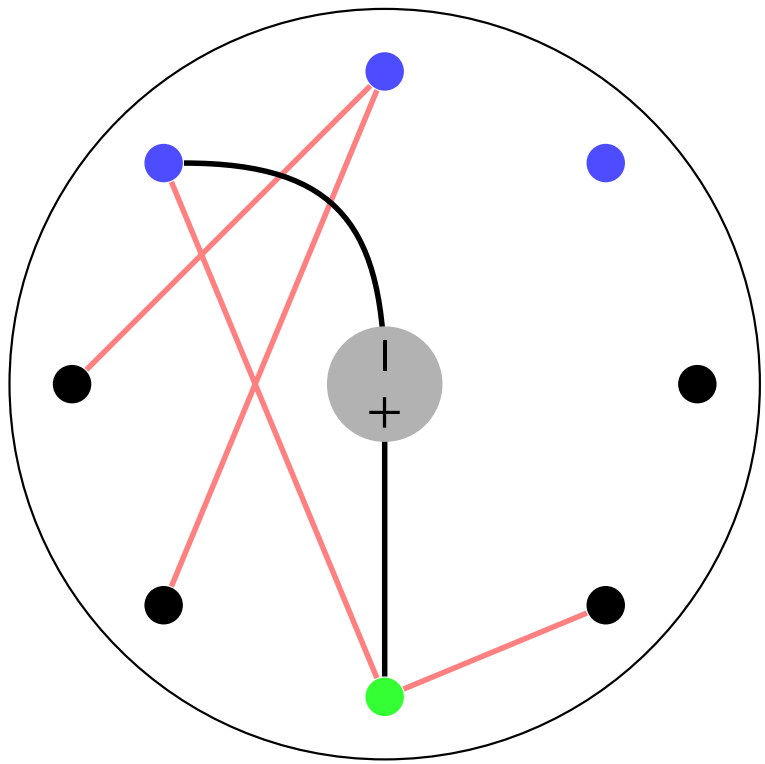

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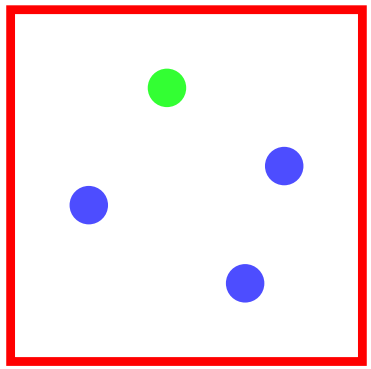

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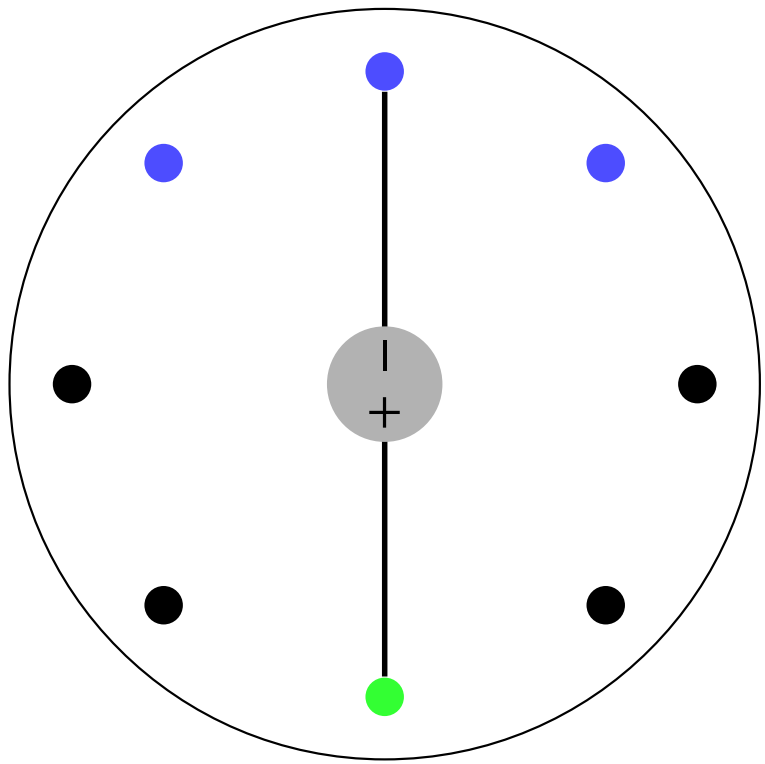

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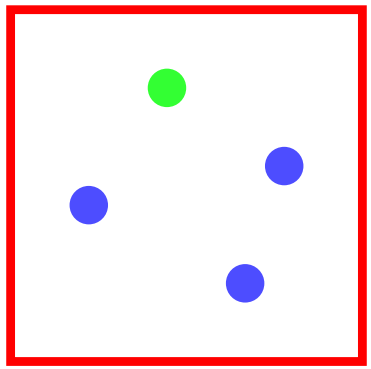

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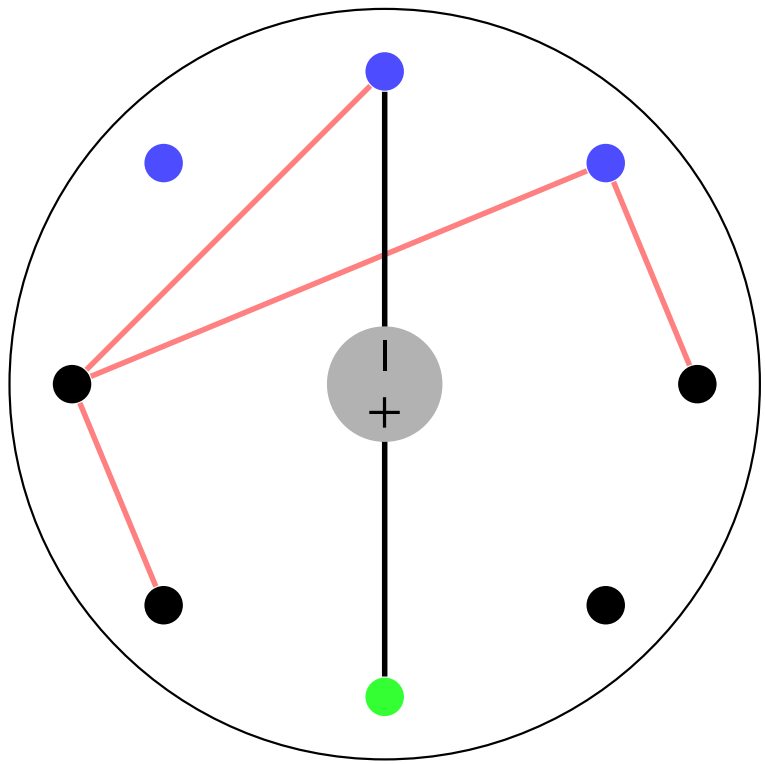

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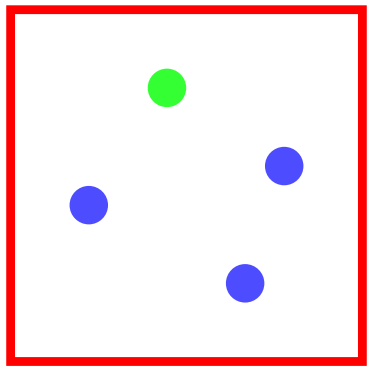

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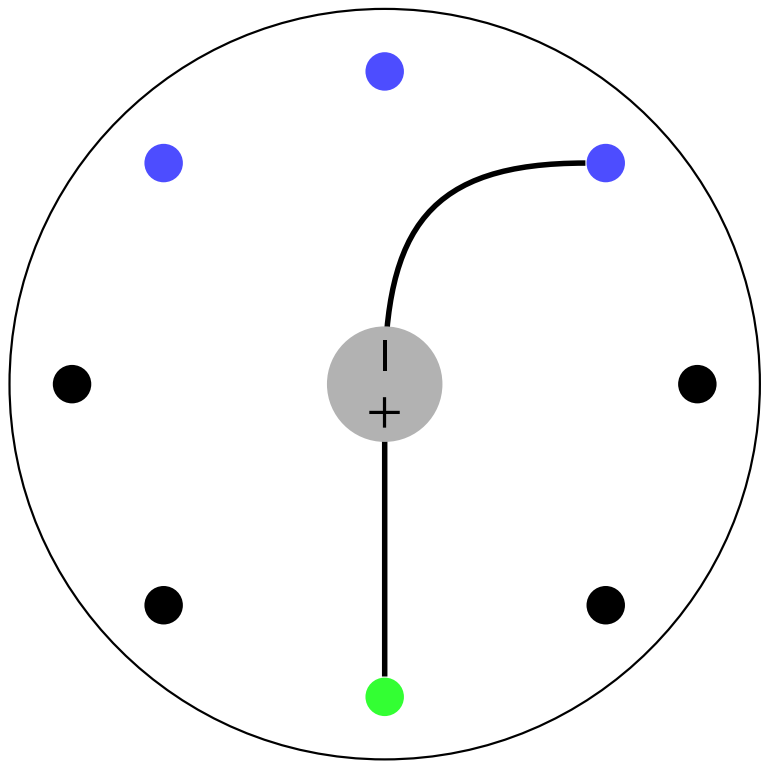

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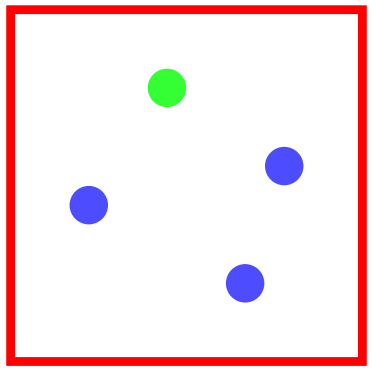

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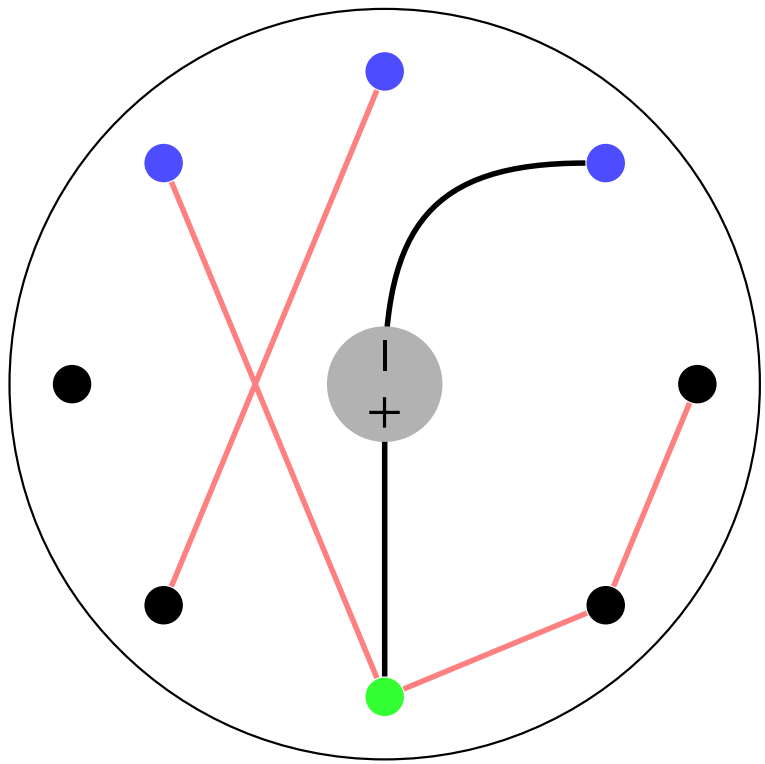

Partitioning via Locality Sensitive Hashing

Resistive embedding maps v to $\mathbb{R}^{\binom{n}{2}}$:

$$y \to \mathbf{B}L^{+}\mathbf{1}_{y}$$

Then

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Final algorithm*:

1. Construct resistive embedding (after dimensionality reduction)
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Set 'near' distance to ≈ 1: a heavy edge is likely to not be cut, but diameter bounded by $\log^{O(1)} n$

Fast recovery for matrices+implications for numerical linear algebra and optimization?

Remarks, related results, open problems

Weighted graphs?

Can one support turnstile updates to the weights?

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Chen, Khanna, Li'22. – lower bound of $\Omega(n^{21/20})$ for vertex incidence sketches!

Spanners?

A subgraph H is a t-spanner of G if for every $u, v \in V$ one has

$$d_G(u,v) \leq d_H(u,v) \leq t \cdot d_G(u,v)$$

Every graph G on n vertices admits a $(2k-1)$-spanner of size $\approx n^{1+1/k}$ (optimal assuming the Erdős girth conjecture).

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A $\widetilde{O}(n^{2/3})$-spanner in $\widetilde{O}(n)$ space

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Single pass sketching algorithm?

Theorem (Fitser-K.-Nouri'21)

A $\widetilde{O}(n^{2/3})$-spanner in $\widetilde{O}(n)$ space(+space/stretch tradeoffs).

Algorithm: take any spectral sparsifier, drop weights on edges.

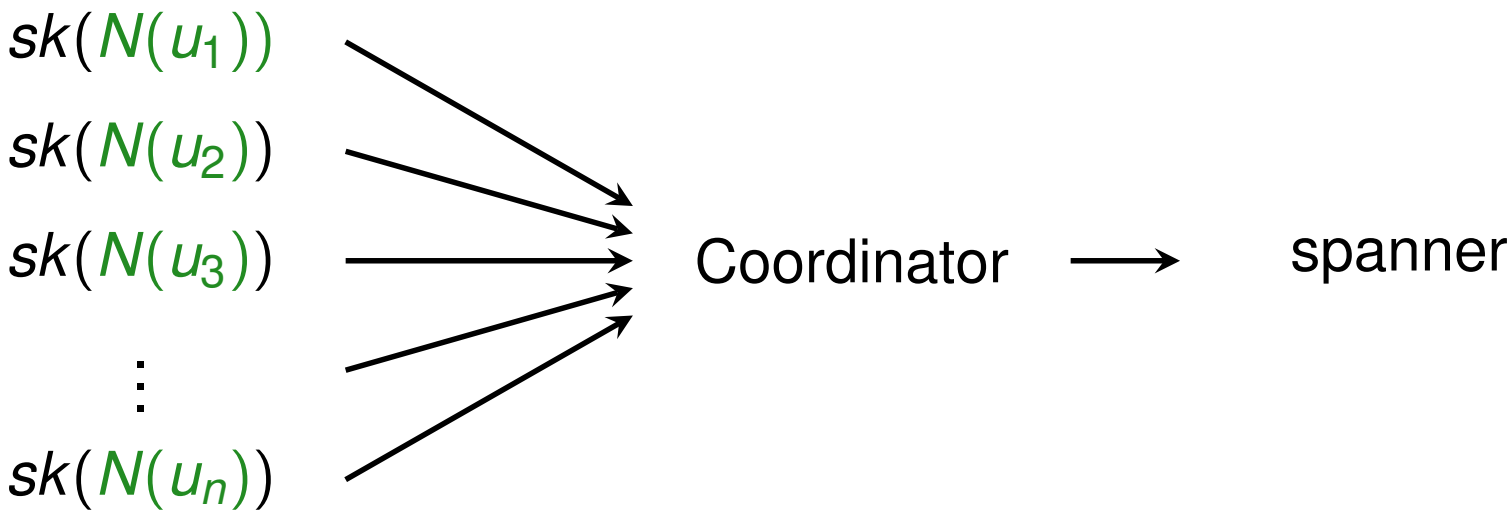
Lower bounds

Input: Each $u \in V$ holds list of its neighbors $N(u)$

Each $u \in V$ sends a sketch of $N(u)$ to coordinator

(assume shared randomness)

Output: $n^{2/3-\Omega(1)}$-spanner of G



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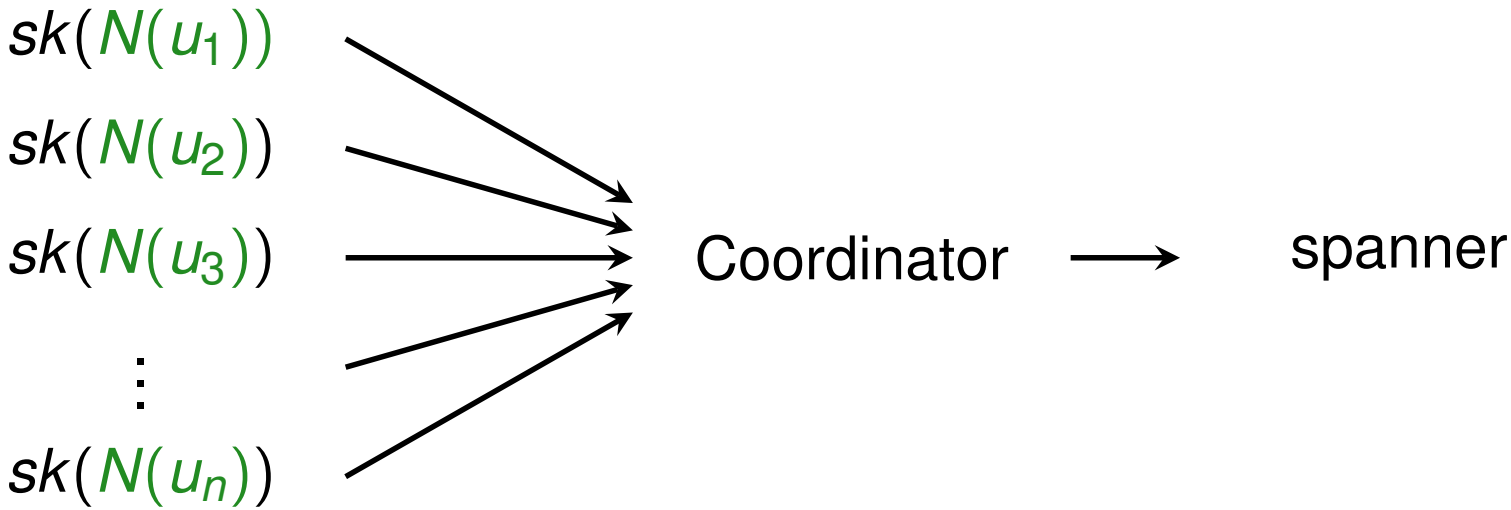

Show that $n^{\Omega(1)}$ communication per node is needed?

Recent progress:Assadi-K.-Yu'23 prove that Filtser-K.-Nouri'21 tradeoffs are optimal for a natural class of sketches

Expander decompositions in sketching

Filtser-K.-Makarov'23 – expander decompositions by sketching

Filtser-K.-Makarov'23, Chen-K.-Makarov-Mazzali'?? – sparsity-independent expander decompositions!

Turn sketching algorithms into low space dynamic algorithms?

Prove that decoded solution is ‘stable’?