Approximating Precedence-Constrained Single Machine Scheduling by Coloring

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Abstract. This paper investigates the relationship between the dimension theory of partial orders and the problem of scheduling precedence-constrained jobs on a single machine to minimize the weighted completion time. Surprisingly, we show that the vertex cover graph associated to the scheduling problem is exactly the graph of incomparable pairs defined in dimension theory. This equivalence gives new insights on the structure of the problem and allows us to benefit from known results in dimension theory. In particular, the vertex cover graph associated to the scheduling problem can be colored efficiently with at most k colors whenever the associated poset admits a polynomial time computable k-realizer. Based on this approach, we derive new and better approximation algorithms for special classes of precedence constraints, including *convex bipartite* and *semi-orders*, for which we give $(1+\frac{1}{3})$-approximation algorithms. Our technique also generalizes to a richer class of posets obtained by lexicographic sum.

1 Introduction

We consider the problem of scheduling a set $N = \{1, \ldots, n\}$ of n jobs on a single machine, which can process at most one job at a time. Each job j has a processing time p_j and a weight w_j, where p_j and w_j are nonnegative integers. We only consider *non-preemptive* schedules, in which all p_j units of job j must be scheduled consecutively. A *partially ordered set* (or *poset*) is a structure $\mathbf{P} = (X, P)$ consisting of a *ground set* X and a *partial order*, i.e. a reflexive, antisymmetric, and transitive binary relation P on X. Jobs have precedence constraints between them that are specified in the form of a poset $\mathbf{P} = (N, P)$, where $(i, j) \in P$ $(i \neq j)$ implies that job i must be completed before job j can be started. The goal is to find a schedule which minimizes the sum $\sum_{j=1}^{n} w_j C_j$, where C_j is the time at which job j completes in the given schedule. In standard scheduling notation (see e.g. Graham et al. [9]), this problem is known as $1|prec|\sum_j w_j C_j$.

The general version of $1|prec|\sum_j w_j C_j$ was shown to be strongly NP-hard by Lawler [13] and Lenstra & Rinnooy Kan [14]. While currently no inapproximability result is known (other than that the problem does not admit a fully polynomial time approximation scheme), there are several polynomial

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time 2-approximation algorithms [17, 20, 10, 3, 2, 15, 1]. For the general version of $1|prec|\sum_j w_j C_j$, closing the approximability gap is considered a longstanding open problem in scheduling theory (see e.g. [21]).

Due to this difficulty, more attention has recently been given to special classes [24, 12, 4, 1]. With this aim, it is worth mentioning that Woeginger [24] proved that the general case of $1|prec|\sum_j w_j C_j$ is not harder to approximate than many fairly restricted special cases, among them the case where all job weights are one. However, for a few relevant special posets with "nice" structural properties, one can obtain better approximation ratios than 2. For the special cases of *interval order* and *convex bipartite* precedence constraints, Woeginger [24] developed polynomial time approximation algorithms with worst case performance guarantee arbitrarily close to the golden ration $\frac{1}{2}(1+\sqrt{5}) \approx 1.61803$. Recently, Ambühl & Mastrolilli [1] settled an open problem first raised by Chudak & Hochbaum [3] and whose answer was conjectured by Correa & Schulz [4]. The results in [1, 4] imply the existence of an exact polynomial time algorithm for the special case of two-dimensional partial orders, improving on previously known approximation algorithms [12, 4], and generalizing Lawler's exact algorithm [13] for series-parallel orders.

Moreover, the most significant implication in [1] is that problem $1|prec|\sum_j w_j C_j$ is a special case of the weighted vertex cover problem in an undirected graph $G_{CS}(\mathbf{P})$ (see [1, 4] and Section 2) that has a node for each ordered pair (i, j) of jobs $i, j \in N$ with $(i, j) \notin P$ and $(j, i) \notin P$ (we say i and j are *incomparable* and write $i \parallel j$ in P). By using this relationship several previous results for the scheduling problem can be explained, and in some cases improved, by means of the vertex cover theory.

Dimension is one of the most heavily studied parameters of partial orders, and many beautiful results have been obtained (see e.g. [22]). Dushnik & Miller [5] introduced dimension as a parameter of partial orders in 1941. Since that time, many theorems have been developed. The *dimension* of a partial order P is the minimum number of linear extensions which yield P as their intersection. More precisely, if P and Q are two partial orders on the same ground set, we say Q is an *extension* of P if $P \subseteq Q$, and we call Q a *linear extension* of P if Q is a linear order and an extension of P. A *realizer* $\mathcal{R}$ of P is a family of linear extensions of P such that $P = \cap\mathcal{R}$, i.e., for all $x, y \in X$, $(x, y) \in P$ if and only if $(x, y) \in L$ for every $L \in \mathcal{R}$. The *dimension* of $\mathbf{P}$, denoted by $dim(\mathbf{P})$ or $dim(X, P)$, is the smallest k such that there exists a realizer $\mathcal{R}$ of P with cardinality k, i.e., $|\mathcal{R}| = k$ ($\mathcal{R}$ is said to be a *k-realizer*). Obviously, $dim(X, P) = 1$ if and only if P is a linear order. With any finite poset $\mathbf{P}$, we can associate a hypergraph $\mathcal{H}_\mathbf{P}$ so that the dimension of $\mathbf{P}$ is equal to the chromatic number of $\mathcal{H}_\mathbf{P}$ [7, 22]. The vertices of $\mathcal{H}_\mathbf{P}$ are the incomparable pairs in P, and this hypergraph is called the *hypergraph of incomparable pairs*. The edges of size 2 in $\mathcal{H}_\mathbf{P}$ determine an ordinary graph $G_\mathbf{P}$, which is called the *graph of incomparable pairs*. Trotter [22] is a good source for further results involving dimension.

In this paper we continue to investigate the structure of problem $1|prec|\sum_j w_j C_j$. We point out an interesting relationship between the dimension theory of partial orders and problem $1|prec|\sum_j w_jC_j$. More specifically, in Section 3 we show that the vertex cover graph $G_{CS}(\mathbf{P})$ associated to $1|prec|\sum_j w_jC_j$ is exactly the graph of incomparable pairs $G_{\mathbf{P}}$ in dimension theory [7, 22]. This equivalence allows us to benefit from dimension theory. In particular, the chromatic number of $G_{CS}(\mathbf{P})$ is at most k, whenever the dimension of the poset at hand is (at most) k. Hochbaum [11] showed that if a given graph for the vertex cover problem can be colored by using k colors in polynomial time, then there exists a $(2-2/k)$-approximation algorithm for the corresponding weighted vertex cover problem. It follows that there exists a $(2-2/k)$-approximation algorithm for $1|prec|\sum_j w_jC_j$ for all those special classes of precedence constraints that admit a polynomial time computable k-realizer.

By following this general approach, we obtain approximation algorithms for relevant special classes of precedence constraints, such as[1] *convex bipartite* precedence constraints (Sections 4) and *semi-orders* (Section 5), for which we exhibit $(1+\frac{1}{3})$-approximation algorithms that improve previous results by Woeginger [24]. However, the technique in [24] also extends to the case of interval order precedence constraints, for which we prove that our approach cannot yield a better approximation ratio (Section 5).

Our technique also generalizes to a richer class of posets obtained by lexicographic sum. Indeed we show, in Section 6, that the number of colors needed to color the graph of incomparable pairs does not increase under the lexicographic sum. In Section 7 we end up by discussing further posets and pointing out some related interesting open problems.

2 Preliminaries

Problem $1|prec|\sum_j w_jC_j$ was recently proved [1] to be a special case of MINIMUM WEIGHTED VERTEX COVER: Given a graph $G=(V,E)$ with weights w_i on the vertices, find a subset $V'\subseteq V$, minimizing the objective function $\sum_{i\in V'} w_i$, such that for each edge $(u,v)\in E$, at least one of u and v belongs to V'.

This result was achieved by investigating the relationship between several different linear programming formulations and relaxations [18, 3, 4] of $1|prec|\sum_j w_jC_j$, using linear ordering variables δ_{ij}. The variable δ_{ij} has value 1 if job i precedes job j in the corresponding schedule, and 0 otherwise. Correa & Schulz [4] proposed the following relaxation of $1|prec|\sum_j w_jC_j$:

$$
\begin{array}{lll}
\text{[CS-IP]} \quad \min & \sum_{i\|j}\delta_{ij}p_iw_j+\sum_{j\in N}p_jw_j+\sum_{(i,j)\in P}p_iw_j & \\
\text{s.t.} & \delta_{ij}+\delta_{ji}\geq 1 \qquad i\parallel j, & (1)\\
& \delta_{ik}+\delta_{kj}\geq 1 \qquad (i,j)\in P, i\parallel k \text{ and } k\parallel j, & (2)\\
& \delta_{i\ell}+\delta_{kj}\geq 1 \qquad (i,j),(k,\ell)\in P, i\parallel \ell \text{ and } j\parallel k, & (3)\\
& \delta_{ij}\in\{0,1\} \qquad i\parallel j. &
\end{array}
$$

[1] Further special classes of posets can be found in [16, 22].

Note that [CS-IP] can be interpreted as the minimum weighted vertex cover in an undirected graph $G_{CS}(\mathbf{P})$, that has a node for each incomparable pair (i, j) of jobs. Two nodes (i, j) and (k, ℓ) are adjacent if either $j = k$ and $i = \ell$, or $j = k$ and $(i, \ell) \in P$, or $(i, \ell), (k, j) \in P$.

Correa & Schulz [4] conjectured that an optimal solution to $1|prec|\sum_j w_j C_j$ gives an optimal solution to [CS-IP] as well. The conjecture in [4] was recently solved by Ambühl & Mastrolilli [1], who proved that any feasible solution to [CS-IP] can be turned in polynomial time into a feasible solution to $1|prec|\sum_j w_j C_j$ without deteriorating the objective value. It follows that problem $1|prec|\sum_j w_j C_j$ is a special case of the weighted vertex cover problem in the graph $G_{CS}(\mathbf{P})$. We refer the interested reader to [1, 4] for a more comprehensive discussion.

We already mentioned that Hochbaum [11] gave a $(2 - 2/k)$-approximation algorithm for the weighted vertex cover problem, whenever the vertex cover graph is k-colorable in polynomial time. Putting everything together we come up with the following result.

Theorem 1. *[1, 4, 11] Problem $1|prec|\sum_j w_j C_j$, for which the graph $G_{CS}(\mathbf{P})$ is k-colorable in polynomial time, has a polynomial time $(2-2/k)$-approximation algorithm.*

3 Posets: Dimension and Coloring

The aim of this section is to point out the connection between $1|prec|\sum_j w_j C_j$ and the dimension theory of partial orders. For this purpose, we need some preliminary definitions.

Let $\mathbf{P} = (N, P)$ be a poset. We say that the partial order $P^d = \{(x, y) : (y, x) \in P\}$ is the *dual* of P. An *alternating cycle* in (N, P) is a collection of incomparable pairs $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ such that $(y_i, x_{i+1}) \in P$ for all i (modulo k). We associate with $\mathbf{P}$ a hypergraph $\mathcal{H}_{\mathbf{P}} = (V, \mathcal{E})$ defined as follows. The vertex set V of $\mathcal{H}_{\mathbf{P}}$ is the set of incomparable pairs $inc(\mathbf{P}) = \{(x, y) \in X \times X : x||y \text{ in } P\}$, and the edge set $\mathcal{E}$ consists of those subsets of V whose duals form alternating cycles. Let $G_{\mathbf{P}}$ denote the ordinary graph determined by all edges of size 2 in $\mathcal{H}_{\mathbf{P}}$. In the literature [22, 7], $\mathcal{H}_{\mathbf{P}}$ and $G_{\mathbf{P}}$ are referred to as the *hypergraph* and the *graph of incomparable pairs*, respectively, and they play an important role in the understanding of dimension. We recall that the *chromatic number* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$, denoted $\chi(\mathcal{H})$, is the least positive integer t for which there is a function $f : V \to [t]$ so that there is no $\alpha \in [t]$ for which there is an edge $E \in \mathcal{E}$ with $f(x) = \alpha$ for every $x \in E$. The following result associates a poset $\mathbf{P}$ to $\mathcal{H}_{\mathbf{P}}$ so that the dimension of $\mathbf{P}$ is the chromatic number of $\mathcal{H}_{\mathbf{P}}$.

Proposition 1 ([22, 7]). *Let $\mathbf{P} = (N, P)$ be a poset, that is not a linear order. Then $dim(\mathbf{P}) = \chi(\mathcal{H}_{\mathbf{P}}) \geq \chi(G_{\mathbf{P}})$.*

Given a k-realizer $\mathcal{R} = \{L_1, L_2, \ldots, L_k\}$ of $\mathbf{P}$, we can easily color $\mathcal{H}_{\mathbf{P}}$ (and $G_{\mathbf{P}}$) with k colors: color vertex (i, j) with some color c whenever $(j, i) \in L_c$. Observe that if all nodes of a hyperedge are colored by the same color c then the linear extension L_c contains an alternating cycle, which is impossible.

The following proposition is immediate and it can be easily checked. It establishes a strong relationship between the dimension theory and $1|prec|\sum_j w_j C_j$.

Proposition 2. *The vertex cover graph $G_{CS}(\mathbf{P})$associated to $1|prec|\sum_j w_j C_j$ and the graph of incomparable pairs $G_{\mathbf{P}}$ coincide.*

A large amount of combinatorial theory exists for posets. Tapping this source can help in designing approximation algorithms.

Theorem 2 ([22, 7]). *Let $\mathbf{P} = (N, P)$ be a poset, that is not a linear order. Then the graph $G_{\mathbf{P}}$ is bipartite if and only if $dim(\mathbf{P}) = 2$.*

Theorem 2 is a well-known result in dimension theory. Correa & Schulz [4] rediscovered it for the vertex cover graph $G_{CS}(\mathbf{P})$, unaware of the connection pointed out by Proposition 2. What is more, the following theorem follows easily from Theorem 1 and Propositions 2 and 1.

Theorem 3. *Problem $1|prec|\sum_j w_j C_j$, whenever precedence constraints are given by a k-realizer, has a polynomial time $(2 - \frac{2}{k})$-approximation algorithm.*

A natural question is for which posets one can construct a k-realizer in polynomial time. In the general case, Yannakakis [25] proved that determining whether the dimension of a poset is at most k is NP-complete for every $k \geq 3$. However, for several special cases, including convex bipartite orders (Section 4) and semi-orders (Section 5), a minimal realizer can be computed in polynomial time.

Finally, by Proposition 1, we remark that $dim(\mathbf{P})$ and $\chi(G_{\mathbf{P}})$ are, in general, not the same (see [7] for an example where $dim(\mathbf{P})$ is exponentially larger than $\chi(G_{\mathbf{P}})$). However, it is an immediate consequence of Theorem 2 that $dim(\mathbf{P}) = \chi(G_{\mathbf{P}})$ when $dim(\mathbf{P}) = 3$. Therefore, a 3-realizer for a 3-dimensional partial order P (as in Sections 4 and 5) immediately gives an optimal coloring for $G_{\mathbf{P}}$.

4 Convex Bipartite Precedence Constraints

In this section we consider $1|prec|\sum_j w_j C_j$ for which the precedence constraints form a so called convex bipartite order. For this class of partial orders, we show how to construct a realizer of size 3 in polynomial time. By Theorem 3, this gives a $(1 + \frac{1}{3})$-approximation algorithm.

A *convex bipartite order* $\mathbf{P} = (N = J^- \cup J^+, P)$ is defined as follows.

1. *The set of jobs are divided into two disjoint sets $J^- = \{j_1, \ldots j_a\}$ and $J^+ = \{j_{a+1}, \ldots, j_n\}$, the* minus-jobs *and* plus-jobs*, respectively.*
2. *For every $k = a+1, \ldots, n$ there are two indices $l(k)$ and $r(k)$ with $1 \leq l(k) \leq r(k) \leq a$ such that $(j_i, j_k) \in P$ if and only if $l(k) \leq i \leq r(k)$* (bipartiteness and convexity)*.*

It is not hard to check that convex bipartite orders can be recognized in polynomial time. Moreover, the class of convex bipartite orders forms a proper subset of the class of general bipartite orders, and a proper superset of the class of strong bipartite orders [16]. Lemma 3 states that the class of convex bipartite orders has dimension ≤ 3. This is indeed a tight bound, since a bipartite order $\mathbf{P}$ is 2-dimensional if and only if it is a strong bipartite order [16]. Finally, we observe that $1|prec|\sum_j w_j C_j$ with strong bipartite orders is solvable in polynomial time [1, 4, 16].

In the subsequent, we sometimes stress that a job j_i is a plus- or minus-job by writing j_i^+ and j_i^-, respectively. We also assume, without loss of generality, that the plus-jobs are numbered such that $i < j$ if and only if $l(i) \leq l(j)$ (breaking ties arbitrarily), where $j_i, j_j \in J^+$.

Given a convex bipartite poset $\mathbf{P} = (N, P)$, we partition its incomparable pairs into three sets E_1, E_2, and E_3 (also depicted in Fig. 1). A pair of incomparable jobs $(j_i, j_j) \in inc(\mathbf{P})$ is a member of

E_1 **if** $i > j$ and $j_i, j_j \in J^-$; **else if** $i < j$ and $j_i, j_j \in J^+$; **else if** $j_i \in J^-$ and $j_j \in J^+$.

E_2 **if** $i < j$ and $j_i, j_j \in J^-$; **else if** $j_i \in J^+, j_j \in J^-$ and there exists a $k > i$ such that $(j_j, j_k) \in P$.

E_3 **if** $i > j$ and $j_i, j_j \in J^+$; **else if** $j_i \in J^+, j_j \in J^-$ and $(j_j, j_k) \notin P$ for all $k > i$.

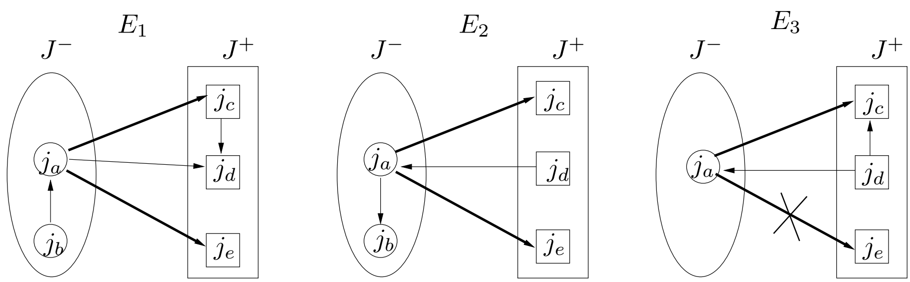

Fig. 1. The round and square nodes correspond to minus-jobs and plus-jobs, respectively. Bold edges correspond to precedence constrains, whereas the other edges are between incomparable jobs. In this example we assume that $a < b$ and $c < d < e$.

The following lemma is a direct consequence of the definition of E_1, E_2, and E_3.

Lemma 1. *Let* $\mathbf{P}$ *be a convex bipartite order then*

1. *The sets* E_1, E_2, *and* E_3 *form a partition of* $inc(\mathbf{P})$*;*
2. *For every* $(i, j) \in inc(\mathbf{P})$, *if* $(i, j) \in E_k$ *then* $(j, i) \notin E_k$, *where* $k \in \{1, 2, 3\}$.

Lemma 2. *Let* $\bar{E}_1 = E_1 \cup P$, $\bar{E}_2 = E_2 \cup P$, *and* $\bar{E}_3 = E_3 \cup P$. *Then* $\bar{E}_1, \bar{E}_2$, *and* $\bar{E}_3$ *are extensions of* P.

Proof. By the definition of $\bar{E}_i$, it follows that if $(j_i, j_j) \in P$ then $(j_i, j_j) \in \bar{E}_i$, where $i = 1, 2, 3$. Moreover, it is easy to see (Fig. 1) that the sets $\bar{E}_1$ and $\bar{E}_3$ do not contain cycles, i.e., are extensions of P.

Now suppose $\bar{E}_2$ contains an alternating cycle C, i.e., it is a non valid extension. By the definition of E_2 we have $C \cap P \neq \emptyset$ and thus $C \cap (J^+ \times J^-) \neq \emptyset$. Let $j_i^- \in J^-$ be the minus-job with largest index in the cycle, i.e., there does not exist a $k > i$ such that $j_k \in J^-$ is part of the cycle. Then $(j_i^-, j_j^+) \in P \cap C$ and $(j_j^+, j_m^-) \in C$ for some jobs $j_j \in J^+$ and $j_m \in J^-$, where $m < i$. However, this implies that there exists an $n > j$ such that $(j_m^-, j_n^+) \in P$ (recall the definition of E_2). Together with convexity and the numbering of plus-jobs this implies $(j_m^-, j_j^+) \in P$, which contradicts the existence of $(j_j^+, j_m^-) \in C$. □

Let L_1, L_2, and L_3 be any linear extensions of $\bar{E}_1, \bar{E}_2$, and $\bar{E}_3$, respectively. That $\mathcal{R} = \{L_1, L_2, L_3\}$ is a realizer follows from the facts that all incomparable pairs are reversed (Lemma 1), and that $\bar{E}_1, \bar{E}_2$, and $\bar{E}_3$ are valid extensions of $\mathbf{P}$ (Lemma 2). Furthermore, all steps involved in creating $\mathcal{R}$ can clearly be accomplished in polynomial time.

Lemma 3. *Given a convex bipartite order* $\mathbf{P} = (N, P)$*, a realizer of size three can be computed in polynomial time.*

Theorem 3 and Lemma 3 give us the following result.

Theorem 4. *Problem* $1|prec|\sum_j w_j C_j$ *for which the precedence constraints form a convex bipartite order has a polynomial time (1+ 1/3)-approximation algorithm.*

5 Interval Orders

A poset $\mathbf{P} = (N, P)$ is an *interval order* [16, 22, 23] if there is a function I assigning to each point $x \in N$ a closed interval $I(x) = I_x = [a_x, b_x]$ of the real line $\mathbb{R}$ so that $(x, y) \in P, x \neq y$ if and only if $b_x < a_y$ in $\mathbb{R}$. The function I is called an *interval representation* of the poset $\mathbf{P}$. Interval orders can be recognized in polynomial time and an interval representation can be computed in $O(n^2)$ time [16].

The best known approximation algorithm for $1|prec|\sum_j w_j C_j$ with interval order precedence constraints is due to Woeginger [24], who gave an (≈ 1.61803)-approximation algorithm. We observe that this ratio can be improved to $(1 + \frac{1}{3})$ in the special case of semi-order precedence constraints. Unfortunately, we show that our techniques do not generalize to interval orders.

5.1 Approximating Semi-orders

A *semi-order*, also called *unit* interval order, has a similar definition as interval orders, but the function I is restricted to only assign unit intervals, i.e., $I(x) = [a_x, a_x + 1]$. Semi-orders can be recognized in $O(n^2)$ time [16, 22]. Moreover, Rabinowitz proved, by constructing a realizer, that the dimension of *semi-orders* is at most three [19, 22]. The constructive proof can easily be turned into a polynomial algorithm and together with Theorem 3, we have the following theorem.

Theorem 5. *Problem* $1|prec|\sum_j w_jC_j$ *for which the precedence constraints form a semi-order has a polynomial time (1+ 1/3)-approximation algorithm.*

5.2 Coloring Interval Orders

For $1|prec|\sum_j w_jC_j$ with interval precedence constraints, one cannot obtain a better than 2-approximation by using our techniques. Indeed we exhibit interval orders where the associated graphs of incomparable pairs have arbitrarily large chromatic number. To prove this, we introduce the *canonical* interval orders. For an integer $n \geq 2$, let $\mathbf{I}_n$ denote the interval order determined by the set of all closed intervals with distinct integer end points from $[n]$. We will find it convenient to view the elements of $\mathbf{I}_n$ as 2-element subsets of $[n]$ with $(\{i_1, i_2\}, \{i_3, i_4\})$ in $\mathbf{I}_n$ if and only if $i_2 < i_3$ in $\mathbb{R}$ or $\{i_1, i_2\} = \{i_3, i_4\}$. The family $\{\mathbf{I}_n : n \geq 2\}$ is called the *canonical* interval orders [23].

Theorem 6. *For any integer* k*, there exists an integer* n_0 *so that if* $n \geq n_0$*, then the chromatic number* $\chi(G_{\mathbf{I}_n})$ *is larger than* k*.*

Proof. The chromatic number $\chi(G_{\mathbf{I}_n})$ is clearly a non-decreasing function of n. We assume that $\chi(G_{\mathbf{I}_n}) \leq k$ for all $n \geq 2$ and obtain a contradiction when n is sufficiently large.

Let the map $\varphi : \binom{[n]}{3} \rightarrow \{1, 2, \ldots, k\}$ denote a coloring of the 3-element subsets of $[n]$. Note that any coloring of $G_{\mathbf{I}_n}$, defines the map φ, by letting $\varphi(\{i, j, k\})$ equal the coloring of the vertex $(\{i, j\},\{j, k\})$ [2] in $G_{\mathbf{I}_n}$.

Let n_0 equal the Ramsey number $R(3 : h_1, h_2, h_3 \ldots, h_k)$, where $h_1 = h_2 = \cdots = h_k = 4$. Now pick n to be greater or equal to n_0 and hence $|[n]| \geq n_0$. Consider any coloring of $G_{\mathbf{I}_n}$ and the corresponding map φ. By Ramsey's Theorem [22], there exists a subset H of $[n]$ with $|H| \geq 4$ so that $\varphi(A) = c$ for every 3-element subset A of H. Consider $\{i, j, k, l\} \subseteq H$, where $i < j < k < l$. We know that $\varphi(\{i, j, k\}) = c$ and $\varphi(\{j, k, l\}) = c$. However, this implies that the adjacent vertices $(\{i, j\},\{j, k\})$ and $(\{j, k\},\{k, l\})$ are colored with the same color. The vertices are adjacent because $\{(\{j, k\}, \{i, j\}), (\{k, l\}, \{j, k\})\}$ forms an alternating cycle.

Thus, for *any* k-coloring, we have two adjacent nodes in $G_{\mathbf{I}_n}$, which are colored by the same color. This contradicts the existence of a valid k-coloring for $G_{\mathbf{I}_n}$ when $n \geq n_0$. □

6 Coloring Lexicographic Sums

So far, we have dealt with some classes of ordered sets and obtained approximation algorithms by coloring. In this section we will ask ourselves how we can use

[2] Note that we can assume without loss of generality that $i < j < k$.

existing posets to build new ordered sets for which the graph of incomparable pairs is still easily colorable. The construction we use here, lexicographic sums, comes from a very simple pictorial idea (see [22] for a more comprehensive discussion). Take a poset $\mathbf{P} = (X, P)$ and replace each of its points $x \in X$ with an ordered set $\mathbf{Q_x}$, the *module*, such that the points in the module have the same relation to points outside it. A more formal definition follows.

Let $\mathbf{P} = (X, P)$ *be a poset, and let* $\mathcal{F} = \{\mathbf{Q_x} = (Y_x, Q_x) : x \in X\}$ *be a family of posets indexed by the elements of* X*. Define the* **lexicographic sum** *of* $\mathcal{F}$ *over* $\mathbf{P}$*, denoted* $\sum_{x\in\mathbf{P}} \mathcal{F}$*, as the poset* $\mathbf{S} = (Z, S)$ *where* $Z = \{z_{xy} : x \in X, y \in Y_x\}$ *and* $(z_{x_1y_1}, z_{x_2y_2}) \in S$ *if and only if both* $x_1 = x_2$ *and* $(y_1, y_2) \in Q_{x_1}$*, or* $(x_1, x_2) \in P$ *(where* $x_1 \neq x_2$*).*

We observe that the resulting class of posets will be a new, larger class than its modules. For example, even if $\mathbf{P}$ and all posets in $\mathcal{F}$ are semi-orders, the lexicographic sum $\sum_{x\in\mathbf{P}} \mathcal{F}$ need not be an interval order: the two-element chain and the two-element antichain both carry semi-orders; Yet the lexicographic sum of two two-element chains over a two-element antichain is the forbidden poset for interval orders [22]. As another example, the lexicographic sum of any 3-irreducible convex bipartite poset and any non-bipartite semiorder poset over a two-element antichain is a poset that is none of the poset previously considered.

A natural question to ask is of course how approximation behaves under lexicographic constructions. With this aim, we prove that the number of colors needed to color the graph of incomparable pairs does not increase under the lexicographic sum. We remark that Hiraguchi (see e.g. [22]) proved that the dimension is "preserved" during lexicographic sum, i.e. $dim\left(\sum_{x\in\mathbf{P}} \mathcal{F}\right) = \max\{dim(\mathbf{P}), \max\{dim(\mathbf{Q_x}) : x \in X\}\}$. However, by Proposition 1 we know that $dim(\mathbf{P})$ and $\chi(G_\mathbf{P})$ are, in general, not the same. This motivates the following result.

Theorem 7. *Let* $\mathbf{P} = (X, P)$ *be a poset and let* $\mathcal{F} = \{\mathbf{Q_x} = (Y_x, Q_x) : x \in X\}$ *be a family of posets. Assume that for each* $i \in \mathcal{P}$*, where* $\mathcal{P} = \{\mathbf{P}\} \cup \mathcal{F}$*, the graph of incomparable pairs* G_i *can be colored with* k_i *colors. Then the graph of incomparable pairs* $G_\mathbf{S}$ *of the lexicographic sum* $\mathbf{S} = \sum_{x\in\mathbf{P}} \mathcal{F}$ *can be colored with* $\max_{i\in\mathcal{P}}\{k_i\}$ *colors.*

Proof. For every $i \in \mathcal{P}$, let C_i be a valid vertex coloring of graph $G_i = (V_i, E_i)$ that uses k_i colors, i.e. a map $C_i : V_i \to \{1, \ldots, k_i\}$ such that $C_i(u) \neq C_i(w)$ whenever u and w are adjacent. Let (z_{ai}, z_{bj}) be any incomparable pair of $G_\mathbf{S}$ and consider the following vertex coloring of $G_\mathbf{S}$:

$$C(z_{ai}, z_{bj}) := \begin{cases} C_\mathbf{P}(a, b) & \text{if } a \neq b; \\ C_{\mathbf{Q_a}}(i, j) & \text{otherwise;} \end{cases} \qquad \text{for all } (z_{ai}, z_{bj}) \in inc(\mathbf{S}). \qquad (4)$$

The claim follows by showing that (4) is a valid coloring of $G_\mathbf{S}$. With this aim it is sufficient to show that for any two adjacent incomparable pairs, namely (z_{ai}, z_{bj}) and $(z_{ck}, z_{d\ell})$, we always have $C(z_{ai}, z_{bj}) \neq C(z_{ck}, z_{d\ell})$. Note that $(z_{ai}, z_{d\ell}) \in P$ and $(z_{ck}, z_{bj}) \in P$, since (z_{ai}, z_{bj}) and $(z_{ck}, z_{d\ell})$ are assumed to be adjacent. We will consider two alternative cases: either we have $a = d$ and $b = c$, or at least one of the previous two conditions is not satisfied, say $a \neq d$, without loss of generality.

(i) $(a = d$ **and** $b = c)$ If $a = b$ then (i, j) and (k, ℓ) are adjacent in $G_{\mathbf{Q_a}}$, and $C(z_{ai}, z_{bj}) = C_{\mathbf{Q_a}}(i, j)$ and $C(z_{ck}, z_{d\ell}) = C_{\mathbf{Q_a}}(k, \ell)$. Otherwise $a \neq b$, and $C(z_{ai}, z_{bj}) = C_{\mathbf{P}}(a, b)$ and $C(z_{ck}, z_{d\ell}) = C_{\mathbf{P}}(b, a)$. The claim follows since $C_{\mathbf{Q_a}}$ and $C_{\mathbf{P}}$ are a valid vertex coloring of $G_{\mathbf{Q_a}}$ and $G_{\mathbf{P}}$, respectively.

(ii) $(a \neq d)$ We start observing that $b \notin \{a, d\}$ by the lexicographic construction. Indeed, by contradiction, if $a = b$ then $(z_{bj}, z_{d\ell}) \in P$ and this, together with $(z_{ck}, z_{bj}) \in P$, implies $(z_{ck}, z_{d\ell}) \in P$; a contradiction since $(z_{ck}, z_{d\ell}) \in inc(\mathbf{S})$. Moreover, if $b = d$ then $(z_{ai}, z_{bj}) \in P$, again a contradiction since $(z_{ai}, z_{bj}) \in inc(\mathbf{S})$. Similarly, we can prove that $c \notin \{a, d\}$. It follows that $C(z_{ai}, z_{bj}) = C_{\mathbf{P}}(a, b)$ and $C(z_{ck}, z_{d\ell}) = C_{\mathbf{P}}(c, d)$. Moreover, since $a \neq d$ we have $(a, d) \in P$. Finally, observe that either $b = c$ or $(c, b) \in P$ and in both cases $C_{\mathbf{P}}(a, b) \neq C_{\mathbf{P}}(c, d)$, and the claim follows. □

A lexicographic sum $\sum_{x \in \mathbf{P}} \mathcal{F}$ is *trivial* if either $\mathbf{P}$ has only one point, or every poset in $\mathcal{F}$ is a one point poset; otherwise the sum is *non-trivial*. A poset is *decomposable* if it is isomorphic to a non trivial lexicographic sum; otherwise it is *indecomposable*. A poset can be decomposed into indecomposable posets in $O(n^2)$ time [16] and by Theorem 7, when coloring, we can restrict our attention on indecomposable posets.

7 Discussion and Open Problems

Semi-Order Dimension. The semi-order dimension of a poset $\mathbf{P} = (X, P)$, denoted $dim_S(\mathbf{P})$, is the smallest k such that there exists k semi-order extensions of P which realize P [6]. Since a linear extension is a semi-order and every semi-order has at most dimension 3 it follows that $dim_S(\mathbf{P}) \leq dim(\mathbf{P}) \leq 3 \cdot dim_S(\mathbf{P})$.

Proposition 3. *Problem* $1|prec|\sum_j w_j C_j$*, where precedence constraints are given as a semi-order realizer of size* k*, has a polynomial time* $(2 - \frac{2}{3k})$*-approximation algorithm.*

Recognizing posets with *interval* dimension 2 can be computed in time complexity $O(n^2)$ [22]. The complexity of recognizing posets with *semi-order* dimension 2 is not known. A polynomial constructive algorithm (constructs the semi-order realizer) would imply a $(1 + \frac{2}{3})$-approximation algorithm for $1|prec|\sum_j w_j C_j$ when precedence constraints form a poset with semi-order dimension at most 2. The class of semi-order dimension 2 posets is a proper superclass of the class of semi-orders and it is not contained in the class of interval orders.

Planar Posets. A poset is planar if its Hasse diagram [22] can be drawn without edge crossings. Our interest in planar posets stems from the fact that a planar poset $\mathbf{P} = (X, P)$ with a greatest or least element has at most dimension 3 [22]. Even though it is NP-complete to recognize if a given partial order is planar [8], we can construct a realizer of size 3 of P in polynomial time if the planar Hasse diagram is given as input [22].

Proposition 4. *Problem* $1|prec|\sum_j w_jC_j$*, where precedence constraints are given as a planar Hasse diagram with a* greatest or least element*, has a polynomial time (1+1/3)-approximation algorithm.*

We also note that planar posets with a greatest *and* least element have at most dimension two. As a consequence they can be recognized in polynomial time and $1|prec|\sum_j w_jC_j$ with precedence constraints of this type can be solved in polynomial time [1, 4]. The situation for planar posets without greatest or least element is more complex, because they can possess arbitrary high dimension [22].

Dimension Approximation. Finally, we remark that the complexity of computing a realizer of a poset is crucial for our approach. At the time being it is an open problem if there is a constant c such that for any partial order of dimension $k \geq 3$, it is possible to construct a realizer of size at most $c \cdot k$ in polynomial time. Any results on this problem would be interesting for the scheduling problem as well as for the dimension theory.

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