Local Algorithms for Finding a Missing String

MISSING STRING: Given a list of M strings of length N ($M < 2^N$), find a string not on the list

Cantor's Diagonal Argument shows:

If $M \leq N$, then we can find a missing string by taking the diagonal of the list

N

M

	1	2	3	...	k	...
x_1	1	0	1		0	
x_2	1	0	1		1	
⋮						
x_k	1	0	0		1	
⋮						

Local Algorithms for Finding a Missing String

MISSING STRING: Given a list of M strings of length N $(M < 2^N)$, find a string not on the list

If the list is truth tables of "easy" functions, we are asking you to find a "hard" function!

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If $M \leq N$, then we can find a missing string by taking the diagonal of the list

N

M

	1	2	3	…	k	…
x_1	0	0	1		0	
x_2	1	1	1		1	
⋮						
x_k	1	0	0		0	
⋮						

For all $i = 1, \ldots, N$, we can obtain the i-th bit of a missing string with ***only one bit probe*** into the input *[probe the i-th bit of the i-th string, output the opposite bit]*

That is, when $M \leq N$, there's a "1-probe" algorithm for MISSING STRING

Under what conditions can we use only k probes? (Why?)

Local Algorithms for Finding a Missing String

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That is, there is a "1-probe" algorithm for MISSING STRING

What can we do with k probes?

Theorem 1 (easy): For $M = kN$, there is **no** $k - 1$ probe algorithm for MISSING STRING.

Proof: Suppose for each $i = 1, \ldots, N$,

your algorithm makes only $k - 1$ probes and computes a missing string y.

The total number of different strings you probed, over all i, is $\leq (k-1)N < M$.

So there's some string on the list you haven't probed at all.

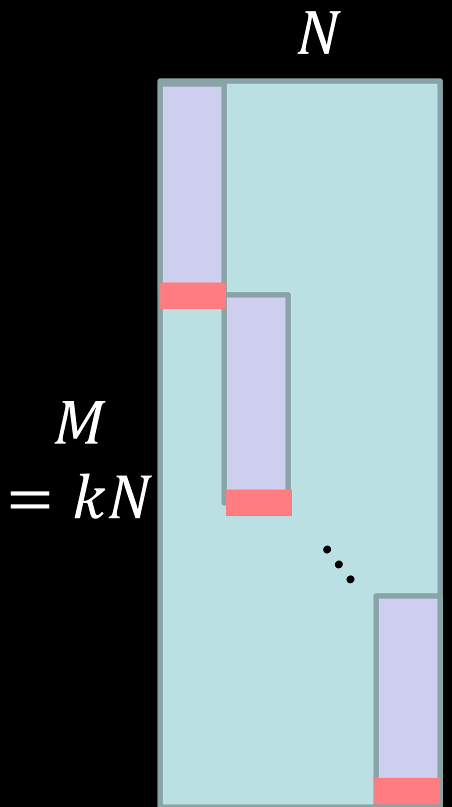
Your algorithm fails on every input that includes y among those non-probed strings.

Local Algorithms for Finding a Missing String

Missing String: Given a list of M strings of length N ($M < 2^N$), find a string not on the list

Theorem 2: For $M \leq kN$, there is an $O(k \log k)$-probe algorithm for Missing String.

Idea: Combine the diagonal argument and the algorithm that makes $O(M)$ probes!



Divide the N bits of each string into $\sim N/b$ blocks of length b

Divide the M strings into $\sim M/t$ blocks of length t, where $t \leq 2^b - 1$.

If $\frac{M}{t} \leq \frac{N}{b}$, then if we find a missing string for each purple block, their concatenation is a missing string for the entire set!

Have: $M = kN$, Want: $M \leq (2^b - 1)\, N/b$. We therefore set:

$$b = \log(k) + \log\log(k) + O(1)$$

Then, to get any particular bit of the missing string, number of probes is

$$O(t) \leq O\left(2^b\right) \leq O(k \log k)$$

What Good are Local Algorithms?

Theorem 2: For $M \leq kN$, there is an $O(k \log k)$-probe algorithm for MISSING STRING.

Corollary: New Time Hierarchy Theorems, against Non-Uniform Programs!

Time Hierarchy Theorem: For "reasonable" g, h where $h(n) >> g(n)$,

TIME($h(n)$) $\not\subseteq$ TIME($g(n)$)

The time hierarchy can be generalized to work for "small non-uniform advice"

Define $f \in$ **TIME[$g(n)$]/$a(n)$** if for every n, there is *some program* of length $a(n)$, running in time $g(n)$, that decides f.

Time Hierarchy Against Advice [Folklore]: For "reasonable" g, h where $h(n) >> g(n)$,

TIME($h(n)$) $\not\subseteq$ TIME($g(n)$)/n

Idea: The hard function running in $O(h(n))$ time treats its input of length n ***as a program,*** and simulates the program on its own code, outputting the opposite answer.

contains uncomputable stuff!

What Good are Local Algorithms?

Theorem 2: For $M \leq kN$, there is an $O(k \log k)$-probe algorithm for MISSING STRING.

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The time hierarchy can be generalized to work for "small non-uniform advice"

Define $f \in$ **TIME[$g(n)$]/$a(n)$** if for every n, there is *some program* of length $a(n)$, running in time $g(n)$, that decides f.

Time Hierarchy Against Advice, Version 2 [Folklore]: For "reasonable" g, h

TIME($2^{n+g(n)} \cdot h(n)$) $\not\subseteq$ TIME($h(n)$)/$g(n)$

Idea: The hard function enumerates all $2^{g(n)}$ programs of length $g(n)$,
and simulates each of them on every possible n-bit input.
Then it computes a missing string from the list of $M = 2^{g(n)}$ strings of length $N = 2^n$

What Good are Local Algorithms?

Theorem 2: For $M \leq kN$, there is an $O(k \log k)$-probe algorithm for MISSING STRING.

Corollary: New Time Hierarchy Theorems, against Non-Uniform Programs!

Old Time Hierarchies Against Advice: For "reasonable" g, h where $h(n) >> g(n)$,

$\mathsf{TIME}(h(n)) \not\subseteq \mathsf{TIME}(g(n))/(n)$ $\quad$ $\mathsf{TIME}(2^{n+g(n)} \cdot h(n)) \not\subseteq \mathsf{TIME}(h(n))/g(n)$

New Hierarchy Against Advice [informally]: for $g(n) \geq n$, $h(n) \geq n$,

$\mathsf{TIME}(2^{g(n)} \cdot g(n) \cdot h(n)/2^n) \not\subseteq \mathsf{TIME}(h(n))/(g(n))$

Finding a function **not** in $\mathsf{TIME}(h(n))/(g(n))$ amounts to finding a missing string of length $N = 2^n$ from a list of $M = O(2^{g(n)})$ strings [all programs of length $g(n)$] where any particular bit of any particular string can be determined in $O(h(n))$ time.
For $k = 2^{g(n)-n}$, we have $M \leq kN$. Thus we can compute any bit of some missing string, probing $O(k \log k) \leq O\left(2^{g(n)-n} \cdot g(n)\right)$ bit positions, each probe taking $O(h(n))$ time.

What Good are Local Algorithms?

Theorem 2: For $M \leq kN$, there is an $O(k \log k)$-probe algorithm for MISSING STRING.

Corollary: New Time Hierarchy Theorems, against Non-Uniform Programs!

Some New Hierarchies:

$$\text{TIME}(5^n) \not\subseteq \text{TIME}(2^n)/(2n)$$

$$\text{TIME}(n^{2c+1}) \not\subseteq \text{TIME}(n^c)/(n + c\log(n))$$

$$\text{TIME}(2^{cn} \cdot poly(n)) \not\subseteq \text{TIME}(O(n))/(cn + n)$$

Note: All the above hierarchies relativize,
and there is an oracle A such that $\text{TIME}^{\text{A}}[2^{cn}] \subseteq \text{TIME}^{\text{A}}[O(n)]\,/(cn + n)$!!

(Main) Open Problems

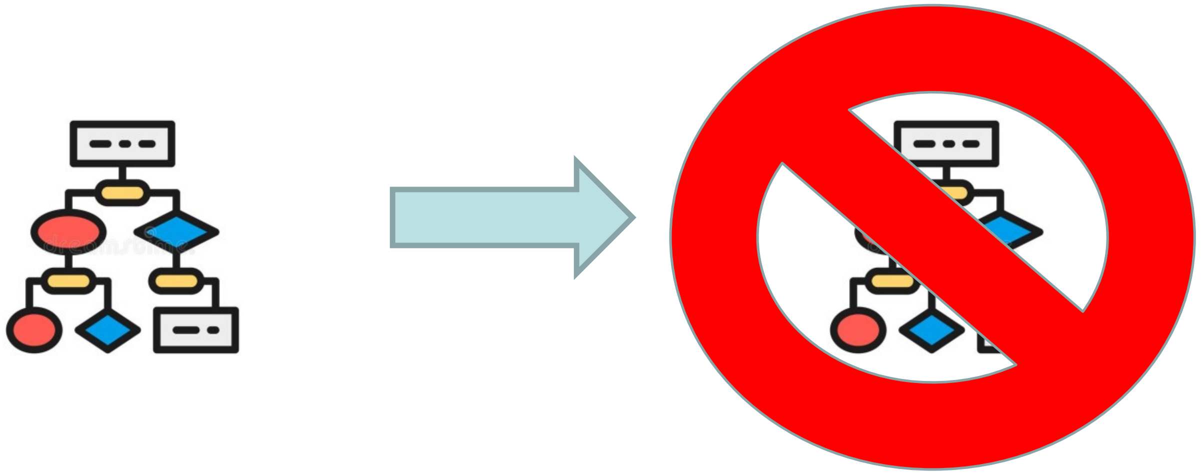
- Does MISSING STRING have small uniform depth-3 circuits, or not?

(it cannot have depth-2 circuits, and it does have depth-4 circuits)

- How many probes (as a function of k) are necessary and sufficient to find a missing string, when $M \leq k\,N$?
  The answer is somewhere between k and $O(k \log k)$

- More connections between MISSING STRING and lower bounds?

How to Prove Lower Bounds With Algorithms



Lecture 4: The Power of Constructing Bad Inputs

(based on work with Lijie Chen, Ce Jin, and Rahul Santhanam, FOCS 2022)

Lower Bounds are Hard to Prove

There are many barriers ⚠️🚫⛔

- Relativization [Baker-Gill-Solovay, 70's]
- Natural Properties [Razborov-Rudich, 90's]
- Algebrization [Aaronson-Wigderson, 00's]

Summary: The standard methods that we use to reason about generic computation cannot resolve P $\neq$ NP (or P $\neq$ PSPACE, or EXP $\neq$ ZPP, or NEXP $\neq$ BPP, etc.)

We apparently know a lot about what strong lower bounds CANNOT look like. We know many limitations on how such proofs must proceed.

What *could* a proof of P $\neq$ NP look like?

We seem to know a lot about what a proof of P $\neq$ NP (and P $\neq$ PSPACE, EXP $\neq$ ZPP, NEXP $\neq$ BPP, etc.) *cannot* look like…

… can we identify properties that such lower bound proofs *must* possess?

What properties are missing from lower bounds we know how to prove, which we will *have* to include in a proof of NEXP $\neq$ BPP (or any of the above)?

What *could* a proof of P $\neq$ NP look like?

Let $f:\{0,1\}^* \to \{0,1\}$ and let $\boldsymbol{\mathcal{A}}$ be a class of algorithms.

A lower bound "$f \notin \boldsymbol{\mathcal{A}}$" is a claim of the form:

$$(\forall A \in \boldsymbol{\mathcal{A}})(\exists \infty\, \boldsymbol{n})(\exists \boldsymbol{x_A} \in \{\mathbf{0},\mathbf{1}\}^{\boldsymbol{n}})[A(\boldsymbol{x_A}) \neq \boldsymbol{f}(\boldsymbol{x_A})]$$

Fix a lower bound problem $f \notin \boldsymbol{\mathcal{A}}$, and fix an algorithm A.
What is the *complexity* of constructing a "bad" x_A of length n?

The literature on lower bounds gives roughly two types of answers:

1. **"Random" or non-constructive ways of choosing x_A**
   Proofs relying on counting/information-theoretic arguments
2. **"Efficient" ways of choosing x_A**
   Proofs based on diagonalization arguments

What *could* a proof of P ≠ NP look like?

Our starting inspiration is from [Gutfreund-Shaltiel-Ta Shma'05]

If $P \neq NP$, then

"bad inputs to SAT algorithms can be efficiently constructed".

The following theorem can be derived from their paper:

For every n^k-time algorithm A,
there is an algorithm R_A that for *infinitely many* n, $R_A(1^n)$ outputs a formula F_n of length n such that F_n is SAT $\Leftrightarrow A(F_n) = 0$.

refuter

[K'00]

Furthermore, R_A runs in $n^{O(k^2)}$ time.

Gutfreund-Shaltiel-Ta Shma '05

If $P \neq NP$, then for infinitely many input lengths, "bad inputs to SAT algorithms can be efficiently constructed"

Let's start by getting a refuter for an n^k-time algorithm A trying to **print** SAT assignments, when they exist. (It first attempts to print a SAT assignment, and outputs UNSAT if that assignment fails to satisfy.)

All bad inputs are satisfiable formulas on which A prints UNSAT

$R_A(1^n)$: Construct a formula F_n encoding the property:

$n^{O(k)}$ **size**

$(\exists\, G, |G| = n$ and assignment $a)[G(a) = 1 \wedge A(G)$ outputs UNSAT]

Run A on F_n. If A prints some G', then output G' else output F_n

We are asking A to print its own counterexamples G'. If A does this for ∞ many n, then we are done. If A does not do this for ∞ many n, then $\{F_n\}$ is printed for almost every n. This set contains ∞ many counterexamples, because we assumed $P \neq NP$.

Gutfreund-Shaltiel-Ta Shma '05

If $P \neq NP$, then for infinitely many input lengths,
"bad inputs to SAT algorithms can be efficiently constructed"

Getting a refuter for an n^k-time algorithm A trying to **decide** SAT:

$R_A(1^n)$: Construct a formula F_n encoding the property:
$(\exists\ G, |G| = n$ and assignment $a)[G(a)\ =\ 1 \wedge A(G)$ outputs UNSAT]
Use search-to-decision on A to search for a SAT assignment to F_n.
(Try setting a variable x to 0, then as 1, seeing what A reports…)

Cases: (1) $A(F_n)$ = "UNSAT". Then output F_n
(2) We find a SAT assignment G'. Then output G'
(3) We find a subformula F'' such that $A(F'')$="SAT",
but $A(F''[x = 0]) = A(F''[x = 1])$="UNSAT"

In case (3), A is wrong on at least one of the 3 subformulas!
Can report the set as "bad". (To get an R_A so that only one is reported,
consider three different algs R_A^1, R_A^2, R_A^3 such that R_A^i reports the i-th)

What *could* a proof of P ≠ NP look like?

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The following theorem can be derived from their paper:

For every n^k-time algorithm A,
there is an algorithm R_A that for *infinitely many* n, $R_A(1^n)$ outputs a formula F_n of length n such that F_n is SAT $\Leftrightarrow A(F_n) = 0$.
Furthermore, R_A runs in $n^{O(k^2)}$ time.

refuter

[K'00]

Say there is a P-constructive separation of $f \notin \mathcal{A}$
if for all $\mathcal{A}$-algorithms A, there is a poly-time algorithm R_A which (for ∞ n) given 1^n can output x_A such that $A(x_A) \neq f(x_A)$.

[GST'05] $P \neq NP \implies$ There's a P-constructive separation of SAT $\notin P$

Constructive Separations

There is a P-constructive separation of "$f \notin \mathcal{A}$"
if for all $\mathcal{A}$-algorithms A, there is a poly-time algorithm R_A which (for ∞ n) given 1^n can output x_A such that $A(x_A) \neq f(x_A)$.

[GST'05] $P \neq NP \implies$ There's a P-constructive separation of SAT $\notin P$

Which lower bound problems *require* constructive separations, and which do not?

This is an algorithmic question about the nature of the lower bound. What *algorithms* are implied by separations?

We Argue: Constructive Separations Are Key!

1. Essentially all open separation problems regarding polynomial time *require* constructive separations

Thm: For all $C \in \{P, ZPP, BPP\}$ and $D \in \{NP, PSPACE, EXP, NEXP, \ldots\}$ If $C \neq D$ then there is a C-constructive separation of $C \neq D$.

2. Making many *known* lower bounds constructive, implies major lower bounds!

More examples of how algorithms can imply lower bounds!

Example 1: [Maass 84] PALINDROMES $\notin NTIME1[o(n^2)]$.

Thm: Making Maass' lower bound P-constructive implies $TIME(2^{O(n)})$ contains functions of exponential circuit complexity. "Constructivizing" this lower bound implies universal derandomization! [IW97]

Therefore, "constructivity" is a property we *want* of lower bounds!

2. Making many *known* lower bounds constructive, *requires* resolving other major lower bound problems

Example 2: Randomized streaming algorithm lower bounds.

Many problems (including simple ones like DISJOINTNESS) are well-known to require $\Omega(n)$-space randomized streaming algorithms

Thm: For any $\Pi \in NP$, a P^{NP}-constructive separation of Π from $(\log n)^{\omega(1)}$ space randomized streaming algs implies $EXP^{NP} \neq BPP$.

Example 3: Randomized query lower bounds.

Promise-MAJORITY: distinguish bit strings with $> ½ + \epsilon$ ones from strings with < ½ - ϵ ones. Well-known to require $\Theta\left(\frac{1}{\epsilon^2}\right)$ queries.

Thm: A "uniform AC^0"-constructive separation of Promise-MAJORITY from randomized query algorithms using $o(\frac{1}{\epsilon^2})$ queries and $poly(\frac{1}{\epsilon})$ time implies $P \neq NP$.

Many more results in the paper…

Constructive Separation of P ≠ PSPACE

Thm: **If $P \neq PSPACE$ then there is a P-constructive separation of $P \neq PSPACE$**

Proof Idea: Instead of SAT, look at TQBF (true quant. Boolean formulas)

Getting a refuter for an n^k-time algorithm A trying to **decide** TQBF:

$R_A(1^n)$: Construct a formula F_n encoding the property:

$$(\exists\ QBF\ "(Qx)G(x)",\ |G| = n)$$

where G itself is a QBF, $Q \in \{\exists, \forall\}$

$$[\textbf{either } A((\forall x)G(x)) \neq A(G(0)) \wedge A(G(1))$$
$$\textbf{or } A((\exists x)G(x)) \neq A(G(0)) \vee A(G(1))]$$

A is inconsistent on these three

Cases: (1) $A(F_n)$ = "false". Then output F_n

In general, we just need a "downward self-reduction"

(2) $A(F_n)$ = "true". Try using search-to-decision to find "$(Qx)G(x)$". Either this fails (in which case you get 3 bad formulas) or it succeeds, but then one of $(Qx)G(x), G(1), G(0)$ must be wrong!

Can report set of 3 as "bad". (Can also modify R_A to only report one)

Making One-Tape TM Lower Bounds Constructive?

[Maass 84] **PALINDROMES $\notin \boldsymbol{NTIME1}[\boldsymbol{o}(\boldsymbol{n}^2)]$.**

Thm: Making Maass' lower bound P-constructive implies that E contains functions of circuit complexity $> 2^{\epsilon n}$ for some $\epsilon > 0$

Proof Sketch: We make a nondeterministic one-tape TM M that takes $N^{1+O(\epsilon)}$ time and correctly decides all palindromes of length $N = 2^n$ with circuit complexity $< 2^{\epsilon n}$.

Therefore, any P algorithm that (on 1^n) prints a "bad" input for M must print a ***hard function***. (The consequence follows by padding.)

M guesses a circuit C of size N^{ϵ} encoding the input x, and guesses $|x|$. "Dragging along" the description of C as it reads the bits of x, it verifies the truth table of C equals x. It verifies x is a palindrome by using C to check that the bits of the 2nd half of x match the 1st half, and it uses $|x|$ to determine when to start checking the 2nd half.

Some Open Questions

- **Relativization tells us that "direct" diagonalization-based proof methods are limited…**
  But any major separation against P (or BPP, or ZPP) will require a key property of diagonalization-based proofs:
  ***the ability to efficiently produce "bad" inputs*.**
  What other methods of proof could yield this consequence?
- **What about separations against LOGSPACE? NC? ACC0?**
  E.g. Does NP $\neq$ LOGSPACE imply a constructive separation?
- **Are there *equivalences* between "derandomization" and constructive separations based on the probabilistic method?**

Thank you!