ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercise Set II, Computational Complexity 2018

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 Give a circuit for computing the XOR function of 4 bits. Can you generalize it to n bits? What is the size of the circuit?

2 *(Exercise 6.8 from the textbook)* A language $L \subseteq \{0,1\}^*$ is sparse if there is a polynomial p such that $|L \cap \{0,1\}^n| \leq p(n)$ for every $n \in \mathbb{N}$. Show that every sparse language is in $\mathbf{P}_{/\mathbf{poly}}$.

3 Show that there is a undecidable language L with a linear-sized circuit family.

4 *(half *)* Recall the definition of an Oblivious TM:

> An *Oblivious* Turing machine (OTM) is a machine for which, at every time t, the j:th head is at cell $s_j(t)$ for some function s_j that only depends on the length of the input.

How can any TM be simulated by an OTM?

(We remark that the best simulation only incurs a logarithmic overhead. However, this is rather complex and, here, you do not need to be overly formal and we allow for a moderate polynomial increase in running time and space. In particular, it is enough to simulate a TM with a single tape as any TM can be simulated efficiently with a single tape.)

5 *(half *)* [Nonuniform Hierarchy Theorem] The nonuniform hierarchy theorem says that for every functions $T, T' : \mathbb{N} \to \mathbb{N}$ with $2^n/n > T'(n) > 10T(n) > n$,

$$\mathbf{SIZE}(T(n)) \subsetneq \mathbf{SIZE}(T'(n)).$$

Show the simpler claim that $\mathbf{SIZE}(n) \subsetneq \mathbf{SIZE}(n^2)$.

6 *(half *)* A Turing machine with advice has, for each n, an advice string α_n, which it is allowed to use in its computation whenever the input has size n. Show that $\mathbf{P}_{/\mathbf{poly}} := \cup_c \mathbf{SIZE}(n^c)$ is equivalent to polynomial-time TMs with polynomial size advice.