ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercise Set III, Computational Complexity 2018

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

Randomized complexity classes

1 Show that $\mathbf{RP} \subseteq \mathbf{NP}$.

2 Show that $\mathbf{BPP} \subseteq \mathbf{EXP}$.

Embarrassingly, it is **not** known whether **BPP** is a strict subset of **NEXP** even though it is believed that $\mathbf{BPP} = \mathbf{P}$.

3 Prove that a language L is in **ZPP** iff there exists a polynomial-time probabilistic TM M with outputs in $\{0, 1, ?\}$ such that for every $x \in \{0, 1\}^*$, with probability 1, $M(x) \in \{L(x), ?\}$ and $\Pr[M(x) =?] \leq 1/3$.

4 *(half *)* Show that $\mathbf{ZPP} = \mathbf{RP} \cap \mathbf{coRP}$.

Polynomial hierarchy

5 Show that $\Pi_i^p = co\Sigma_i^p := \{\bar{L} : L \in \Sigma_i^p\}$. Also show that $\Pi_i^p \subseteq \Sigma_i^p$ implies $\Pi_i^p = \Sigma_i^p$.

6 [Each level has a complete problem] Define $\Sigma_i SAT$ to be the language consisting of Boolean formulas φ such that $\exists u_1 \forall u_2 \ldots Q_i u_i \varphi(u_1, u_2, \ldots, u_i) = 1$. Show that $\Sigma_i SAT$ is a complete problem for Σ_i^p.

(One can similarly define $\Pi_i SAT$ that is complete for Π_i^p.)

7 [**PH** is unlikely to have complete problems] Show that if there exists a language L that is **PH** complete, then there exists an i such that $\mathbf{PH} = \Sigma_i^p$, i.e., the hierarchy collapses to its i:th level.

8 Suppose A is some language such that $\mathbf{P}^A = \mathbf{NP}^A$. Then show that $\mathbf{PH}^A \subseteq \mathbf{P}^A$ (in other words, the proof of Theorem 3 from Lecture 6 relativizes).