1 Introduction

- Recall last lecture: Non-determinism.
- Today: why diagonalization alone will not solve $P$ vs $NP$ question (see last lecture’s notes) + beginning of circuits, non-uniform computation.

2 Circuits

A circuit $C$ has $n$ inputs and $m$ outputs, and is constructed with AND, OR, and NOT gates. Each gate has fan-in 2 except the NOT gate which has fan-in 1. The out-degree can be any number. A circuit is not allowed to have any cycles.

**Example 1** A circuit $C$ computing the XOR function, i.e., $C(x_1, x_2) = 1$ iff $x_1 \neq x_2$:

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\begin{center}
\begin{tikzpicture}
    \node (input1) at (0,0) {$x_1$};
    \node (input2) at (1,0) {$x_2$};
    \node (not1) at (0,-1) {$\neg$};
    \node (not2) at (1,-1) {$\neg$};
    \node (and1) at (0,-2) {$\land$};
    \node (and2) at (1,-2) {$\land$};
    \node (or) at (0.5,-3) {$\lor$};

    \draw[->] (input1) -- (not1);
    \draw[->] (input2) -- (not2);
    \draw[->] (not1) -- (and1);
    \draw[->] (not2) -- (and2);
    \draw[->] (and1) -- (or);
    \draw[->] (and2) -- (or);
\end{tikzpicture}
\end{center}
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**Definition 1 (Size)** The size of a circuit $C$, denoted by $|C|$, is the number of its gates.

- The size of the XOR circuit $C$ above is 5.

**Definition 2 (Circuit families and language recognition)** Let $T : \mathbb{N} \to \mathbb{N}$ be a function. A $T(n)$-size circuit family is a sequence of $\{C_n\}_{n \in \mathbb{N}}$ of Boolean circuits, where $C_n$ has $n$ inputs and a single output, and its size $|C_n| \leq T(n)$ for every $n$.

We say that language $L$ is in $\text{SIZE}(T(n))$ if there exists a $T(n)$-size circuit family $\{C_n\}_{n \in \mathbb{N}}$ such that for every $x \in \{0, 1\}^n$, $x \in L \iff C_n(x) = 1$.

**Example 2** For any $B \subseteq \{0, 1\}^*$, the unary language $U_B = \{1^n : \text{exists a string of length } n \text{ in } B\}$ has a linear-sized circuit family. If $1^n \in U_B$ the circuit is simply a tree of AND gates and otherwise if $1^n \notin U_B$ then the circuit $C_n$ is the trivial circuit that always outputs 0.
Example 3  The language \{\langle m, n, m + n \rangle : m, n \in \mathbb{Z}\} also has linear-sized circuits that implement the grade-school algorithm for addition.