1 Last Lecture

In the previous lecture, we introduced the probabilistic method, which allowed us to prove the existence of a combinatorial object with specified properties. In general, the argument goes as follows: select an object at random from the set, and calculate the probability that it satisfies the required property; if this probability is strictly positive then such an object must exist. However this approach is not powerful enough when the events we are considering are not independent.

2 Lovász Local Lemma

Definition 1 An event \( A \) is mutually independent from a set \( \{B_i\} \) of events if for every subset \( \beta \) of events or their complements contained in \( \{B_i\} \)

\[
Pr[A | \beta] = Pr[A]
\]

In a general setting, we have a set of \( n \) bad events \( \{A_i\} \) that we are trying to avoid, such that \( Pr[A_i] \leq p < 1 \), for \( i \in \{1, 2, \ldots, n\} \). If we assume that these events are independent, then their complements are independent as well and we can show that

\[
Pr[\bigwedge_i \overline{A_i}] \geq (1 - p)^n > 0
\]

However, if we remove the independence assumption, the union bound yields

\[
Pr[\bigwedge_i \overline{A_i}] \geq 1 - \sum_i Pr[\overline{A_i}]
\]

The Lovász Local Lemma improves upon the union bound in the case where the events are not mutually independent, but their dependencies are restricted. It was proved by Erdős and Lovász in 1975 [1].

Theorem 2 (Lovász Local Lemma) Let \( A_1, A_2, \ldots, A_n \) be a set of "bad" events with \( Pr[A_i] \leq p < 1 \), and each \( A_i \) is dependent on at most \( d \) other \( A_j \). If \( p \cdot (d + 1) \cdot e \leq 1 \), then

\[
Pr\left[\bigwedge_{i=1}^n \overline{A_i}\right] > 0
\]

Before proving the theorem, we give an example showing that the bound is almost tight. Consider events \( A_1, \ldots, A_{d+1} \), each happening with probability \( \frac{1}{d+1} \). We assume that in each outcome exactly one bad event happens. Consider figure 1. In that case we have \( p \cdot (d + 1) = 1 \) and \( Pr[\bigwedge_{i=1}^n \overline{A_i}] = 0 \).

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<th>( A_1 )</th>
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<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( A_5 )</th>
<th>( A_6 )</th>
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<th>( A_8 )</th>
<th>( A_9 )</th>
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<td>( \frac{1}{10} )</td>
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Figure 1: Example with \( p = Pr[A_i] = \frac{1}{10} \) and \( d = 9 \)
This shows the tightness of the bound. We continue with the proof of the Lovász Local Lemma. To do so, we will use the following lemma:

**Lemma 3** For any $S \subset \{1, \ldots, n\}$ and $i \in \{1, \ldots, n\}$ the following holds:

$$\Pr \left[ A_i \mid \bigwedge_{j \in S} \overline{A}_j \right] \leq \frac{1}{d+1}$$

**Proof** We show the lemma by induction on the size $m$ of $S$.

**Base Case**
For $m = 0$ we obtain

$$\Pr[A_i] \leq p \leq \frac{1}{(d+1) \cdot e} < \frac{1}{(d+1)}$$

where the second inequation follows by the condition of Lovász Local Lemma.

**Inductive Case**
Assume that the claim is true for all $S$ with $|S| < m$. We prove the claim for an $S$ with $|S| = m$. At first we partition $S$ into $S = S_1 \cup S_2$ such that $S_1$ contains those events of $S$ that $A_i$ depends on. Recall that

$$\Pr[A \mid B] = \frac{\Pr[A \land B]}{\Pr[B]}$$

In the following we will apply this equation several times. Using this equation we get

$$\Pr \left[ A_i \mid \bigwedge_{j \in S} \overline{A}_j \right] = \frac{\Pr \left[ A_i \land \bigwedge_{j \in S_1} \overline{A}_j \mid \bigwedge_{j \in S_2} \overline{A}_j \right]}{\Pr \left[ \bigwedge_{j \in S_1} \overline{A}_j \mid \bigwedge_{j \in S_2} A_j \right]}$$

To bound this term further, we want to give an upper bound on the numerator and a lower bound on the denominator. The upper bound on the numerator is easily obtained as:

$$\Pr \left[ A_i \land \bigwedge_{j \in S_1} \overline{A}_j \mid \bigwedge_{j \in S_2} \overline{A}_j \right] \leq \Pr \left[ A_i \mid \bigwedge_{j \in S_2} \overline{A}_j \right] = \Pr[A_i]$$
We continue with the lower bound on the denominator. W.l.o.g. let $S_1 = \{1, \ldots, r\}$. Therefore, we can rewrite the denominator as:

$$\Pr \left[ \bigwedge_{j=1}^r \bar{A}_j \mid \bigwedge_{j \in S_2} \bar{A}_j \right] = \Pr \left[ A_1 \mid \bar{A}_2 \wedge \bar{A}_3 \wedge \cdots \wedge \bar{A}_r \wedge \bigwedge_{j \in S_2} \bar{A}_j \right] \cdot \Pr \left[ \bar{A}_2 \mid A_3 \wedge \bar{A}_4 \wedge \cdots \wedge \bar{A}_r \wedge \bigwedge_{j \in S_2} \bar{A}_j \right] \cdot \cdots \cdot \Pr \left[ \bar{A}_r \mid \bigwedge_{j \in S_2} \bar{A}_j \right]$$

$$= \left( 1 - \Pr \left[ A_1 \mid \bar{A}_2 \wedge \bar{A}_3 \wedge \cdots \wedge \bar{A}_r \wedge \bigwedge_{j \in S_2} \bar{A}_j \right] \right) \cdot \left( 1 - \Pr \left[ A_2 \mid \bar{A}_3 \wedge \bar{A}_4 \wedge \cdots \wedge \bar{A}_r \wedge \bigwedge_{j \in S_2} \bar{A}_j \right] \right) \cdot \cdots \cdot \left( 1 - \Pr \left[ \bar{A}_r \mid \bigwedge_{j \in S_2} \bar{A}_j \right] \right)$$

$$\geq \left( 1 - \frac{1}{d+1} \right) \cdot \left( 1 - \frac{1}{d+1} \right) \cdot \cdots \cdot \left( 1 - \frac{1}{d+1} \right)$$

$$= \left( 1 - \frac{1}{d+1} \right)^r \geq \left( 1 - \frac{1}{d+1} \right)^d > \frac{1}{e}$$

This concludes the proof of the lemma. ■

We will now apply the lemma to prove the theorem.

**Proof** We need to show that desired event occurs with positive probability. This probability can be bounded by

$$\Pr \left[ \bigwedge_{i=1}^n \bar{A}_i \right] = \Pr \left[ A_1 \mid \bar{A}_2 \wedge \bar{A}_3 \wedge \cdots \wedge \bar{A}_n \right] \cdot \Pr \left[ \bar{A}_2 \mid \bar{A}_3 \wedge \bar{A}_4 \wedge \cdots \wedge \bar{A}_n \right] \cdot \cdots \cdot \Pr \left[ \bar{A}_n \right]$$

$$\geq \left( 1 - \frac{1}{d+1} \right)^n > 0$$

where the inequality follows from lemma 3. ■
2.1 Application: $k$-SAT

We use the Lovász Local Lemma to prove that any $k$-CNF formula is satisfiable if a certain condition holds. First we define the $k$-SAT problem:

**Input** Boolean formula $\varphi = \bigwedge_{i=1}^{n} C_i$ in $k$-CNF.

with $m$ boolean variables $x_1, \ldots, x_m$

and $n$ clauses $C_1, \ldots, C_n$ being conjunctions of $k$ different literals

**Output** Decide whether there is a satisfying interpretation of $\varphi$

We formalize the condition to guarantee the satisfiability of an $k$-CNF formula.

**Lemma 4** Any instance $\varphi$ of $k$-Sat in which no variable appears in more than $\frac{2^{k-2}}{k}$ clauses is satisfiable.

**Proof** We take an assignment $X : \{x_1, \ldots, x_m\} \rightarrow \{true, false\}$ uniformly at random. Let $A_i$ be the "bad" event that clause $C_i$ is unsatisfied by the random assignment $X$.

Let us determine the probability $\Pr[A_i]$. Since each clause consists of $k$ different literals and each literal evaluates to false with probability $\frac{1}{2}$ we get:

$$\Pr[A_i] = \left(\frac{1}{2}\right)^k =: p$$

Now let us bound the maximum dependency of $A_i$. Let $j \neq i$. It is obvious that $A_i$ is dependent on $A_j$ if and only if $C_i$ and $C_j$ share a variable. Each variable may occur in at most $\frac{2^{k-2}}{k}$ clauses. Since $C_i$ consists of $k$ literals, it can be dependent on at most $k \cdot \frac{2^{k-2}}{k} = 2^{k-2}$ other clauses.

Thus we can apply the Lovász Local Lemma with $p = \frac{1}{2^k}$ and $d = 2^{k-2}$ and get $p \cdot (d+1) \cdot e \leq p \cdot d \cdot 4 = 1$ for sufficiently large $d$. Applying the Lovász Local Lemma we get that the probability of getting a satisfying assignment $\Pr[\bigwedge_{i=1}^{n} \neg A_i] > 0$. Using the probabilistic method, we can conclude that there is in fact a satisfying assignment for $\varphi$.

2.2 Exercise 1: Application: 2-COLORING in a hyper graph

We use Lovász Local Lemma to prove that certain hyper graphs always have a 2-Coloring. A hyper graph is a tuple $H = (V, E)$ with vertices $V$ and hyper edges $E$. That is, each edge is a subset of the set of vertices $V$. A 2-Coloring is an assignment $c : V \rightarrow \{\text{red, blue}\}$ of colors to vertices such that there is no monochromatic hyper edge. A monochromatic edge is an edge connecting only nodes of the same color.

**Exercise 1** Let $H = (V, E)$ be a hyper graph in which every edge has at least $k$ elements and intersects at most $d$ other edges. For which $k, d$ has $H$ a 2-Coloring?

**Solution** We take a coloring uniformly at random. Let $A_e$ be the bad event that edge $e$ became monochromatic. This is the case when all vertices of $e$ are either blue or all red. Now consider the probability of $A_e$:

$$\Pr[A_e] = \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k = 2^{1-k} =: p$$

We need to choose $d$ such that the condition of the Lovász Local Lemma is satisfied. That is, $p \cdot (d+1) \cdot e \leq 1$. This can be achieved by choosing $d \leq \frac{2^{1-k}}{e} - 1$. 

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2.3 Exercise 2: Subtrees

The proof of Lovász Local Lemma given beforehand was non-constructive. Later on we will give a constructive proof. To do so, the following statement will be useful.

**Exercise 2** Consider a graph \( G \) of degree at most \( d + 2 \). Give an upper bound on the number of subtrees of \( G \) consisting of \( s \) nodes.

**Solution** For a fixed root we order the vertices in an arbitrarily (but fixed) order. Note, that this also orders the neighbors of a vertex uniquely. We will encode each tree with a bitstring of length \( (d + 1) \cdot s \) containing \( s - 1 \) ones. We count the number of trees rooted at a vertex \( v \).

Assume we had such an encoding. Then, the number of trees can be at most the number of bitstrings of this kind. That is:

\[
\left( \frac{(d + 1) \cdot s}{s - 1} \right) \leq \left( \frac{(d + 1) \cdot s \cdot e}{s - 1} \right)^s \leq (d + 2) \cdot e\]

It remains to be shown how a tree \( T \) can be encoded in such a fashion. The general idea is to do a pre-order traversal of \( T \) starting at the root \( v \). When we discover a new vertex \( u \), we output a string of length \( (d + 1) \). This string indicates whether for every neighbor \( w \) of \( u \), \( w \) is contained in the tree. In particular, the string has a ”1” at the \( i \)th position if and only if, the \( i \)th neighbor is a child of \( u \) in \( T \). This is visualized in figure 2:

![Figure 2: Tree of size 4 with coding 1100 0000 1000 0000](image)

Because we have \( n \) possible roots, the total number of possible trees can be bounded by \( n \cdot ((d+2)\cdot e)^s \).
2.4 A Constructive Proof of Lovász Local Lemma

The previous proof in this lecture was non-constructive. It did not show how to obtain an algorithm that computes the desired object. In this subsection we present the constructive version of this proof in [2] and [3] by giving a randomized algorithm for $k$-SAT. For the general case a similar algorithm can be constructed. We use the notation as before.

**Algorithm 1** Randomized $k$-SAT

1: Pick random truth assignment $X : \{x_1, \ldots, x_m\} \rightarrow \{true, false\}$
2: while there is an unsatisfied clause $C$ do
3:    Reflip all variables in $C$
4: end while
5: Return the satisfying assignment

Obviously this algorithm returns a satisfying assignment on termination. It remains to be shown that it in fact terminates in acceptable runtime.

**Theorem 5** Let $p \cdot (d + 1) \cdot e \leq 1 - \varepsilon$ for a constant $\varepsilon > 0$. Then the expected amount of reflips in the algorithm will be in $\Theta(n)$.

**Proof** Let $C_1, C_2, \ldots$ be the (possibly infinite) sequence of clauses that are reflipped by the algorithm in that order. With each $C_i$ we associate the tree $T_i$ with a root node labeled as $C_i$. We add clause $C_1, \ldots C_{t-1}$ in reversed order as follows: For $i = t - 1, \ldots, 1$ we ignore $C_i$ if it does not share any variable with any clause in $T_t$. Otherwise, we add $C_i$ as a child of the deepest clause sharing at least one variable with $C_i$.

![Figure 3: Example with reflipped clauses](image)

We use the constructed tree $T_i$ to prove the theorem by bounding the expected number of reflips. To do so, we have to measure the probability of a certain tree appearing.
Claim  Consider any tree $T_t$ of $s$ nodes. The probability of this tree appearing is at most $2^{-sk}$.

Proof  Consider a leaf of maximal depth in $T_t$. The corresponding clause does not share any variables with a clause which has been reflipped earlier. Thus, this clause was unsatisfied by the original assignment. This happens with probability $2^{-k}$. Since every variable in this clause is reflipped, the assignment of variables of higher clauses is independent. Hence, each node independently appears with probability $2^{-k}$. Because the tree consists of $s$ nodes, its probability of appearing is $2^{-sk}$. ■

We use the claim to calculate the estimated amount of reflips for finding a satisfying assignment.

$$
\mathbb{E}[\#\text{reflips}] = \mathbb{E}[\# \text{trees appearing}]
= \sum_{s=1}^{\infty} \mathbb{E}[\# \text{trees of size } s]
\leq n \cdot \sum_{s=1}^{\infty} \left( (d+1) \cdot e \right)^s \cdot 2^{-sk}
= n \cdot \sum_{s=1}^{\infty} \left( (d+1) \cdot e \cdot 2^{-k} \right)^s
= n \cdot \sum_{s=1}^{\infty} \left( (d+1) \cdot e \cdot p \right)^s
\leq n \cdot \sum_{s=1}^{\infty} (1 - \varepsilon)^s
\leq \frac{n}{1 - (1 - \varepsilon)} = \frac{n}{\varepsilon} = \Theta(n)
$$

This finishes the proof of the theorem. ■

References

