Topics in Theoretical Computer Science	April 13, 2015

Lecture 8

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These notes are partially based on [1] and [2].

1 Introduction

In the previous lectures, we have been studying about matroids. We saw that given a matroid $M = (E, \mathcal{I})$, we have a greedy algorithm to find a maximum weight independent set. We also gave a characterization for the matroid polytope, which is defined as follows

Definition 1 (Matroid polytope) *Given a matroid $M = (E, \mathcal{I})$, its matroid polytope $P_M = conv(\{x_s \in \{0,1\}^{|E|} : s \in \mathcal{I}\})$ is the convex hull of the incidence vectors of the independent sets of M. We also showed that the matroid polytope can be described as the set of inequalities $P_M = \{\mathbb{R}_+{}^{|E|} : \sum\limits_{e \in S} x_e \leq r(S) \forall S \subseteq E\}$.*

In the previous lecture, we saw what is matroid intersection, which is defined as follows:

Definition 2 (Matroid intersection) *Given two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ over the same ground set E, the intersection of two matroids is defined as $M_1 \cap M_2 = (E, \mathcal{I}_1 \cap \mathcal{I}_2)$.*

Note that a matroid intersection may not be a matroid in general. We also proved the matroid intersection theorem, which says the following:

Theorem 3 (Matroid intersection theorem) *For matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$*

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq E} (r_1(U) + r_2(E \backslash U))$$

We proved this by giving a polynomial time algorithm to find an independent set of maximum cardinality in $\mathcal{I}_1 \cup \mathcal{I}_2$ by using 'Augmenting paths'.

The main aim of this lecture is to prove what is called the **Matroid intersection polytope theorem**. This theorem says that the matroid intersection polytope of $M_1 \cap M_2$ is nothing but the intersection of the matroid polytopes of M_1 and M_2.

1.1 Preliminaries

Before we go on to define the matroid intersection polytope and make the matroid intersection polytope theorem formal, we summarize a few things about linear program polytopes that we will use in this lecture and also give some definitions. We observe that we have the following for a bounded polytope P:

- If P is non-empty, it has extreme points.
- Suppose $x \in P$, then for any cost vector c, there exists an extreme point $x*$ such that $c(x*) \leq c(x)$.
- Every extreme point in P is an optimal solution to some cost function.

Tight constraints of an extreme point: In an n-dimensional linear program every extreme point is a unique solution to a linear system given by n linearly independent tight constraints.
Finally we define two operations on matroids, namely deletion and contraction which give us a modified matroid.

Definition 4 (Deletion) *Given a matroid $M = (E, \mathcal{I})$ with $e \in E$, 'deleting' e gives us*

$$M \backslash e = (E - e, \mathcal{I}')$$

where

$$\mathcal{I}' = \{I \subseteq E - e : I \in \mathcal{I}\}$$

In a graphic matroid this is equivalent to deleting an edge.

Definition 5 (Contraction) *Given a matroid $M = (E, \mathcal{I})$ with $e \in E$, 'contracting' e gives us*

$$M/e = (E - e, \mathcal{I}')$$

where

$$\mathcal{I}' = \{I \subseteq E - e : I + e \in \mathcal{I}\} \qquad if \quad e \in \mathcal{I}$$

If $e \notin \mathcal{I}$ we have $M/e = M\backslash e$. In a graphic matroid this is equivalent to contracting an edge.

Before moving on to the main part, we solve a few exercises to revise some concepts about matroids learnt in the previous lectures.

Exercise 1 Let $G = (V_L \cup V_R, R)$ be a k-regular bipartite graph with edge weights $w : E \to \mathbb{R}$. Prove that G has a perfect matching whose cost is no more than $\frac{1}{k} \sum\limits_{e \in E} w(e)$.

Proof We consider the perfect matching polytope P for the graph G. We prove the desired result using the fact that if $x \in P$, then for any cost vector c, there exists an extreme point x^* such that $c(x^*) \leq c(x)$. For this, consider $x \in \mathbb{R}^{|E|}$ such that $x_e = \frac{1}{k} \forall e \in E$. It is easy to see that x is a feasible solution to P, that is $x \in P$(we also used this fact in the solution to the last problem of homework assignment 2). We consider the cost vector as the weight of a solution. For some $y \in \mathbb{R}^{|E|}$, its weight is $W(y) = \sum\limits_{e \in E : y \in \mathbb{R}^{|E|}} y_e w(e)$. Since $x \in P$ we obtain the fact that there is an extreme point $x^* \in \mathbb{R}^{|E|}$ of P such that $W(x^*) \leq W(x)$ which gives us.

$$W(x^*) \leq \sum_{e \in E} x_e w(e) = \frac{1}{k} \sum_{e \in E} w(e)$$

The inequality $W(x^*) \leq \frac{1}{k} \sum\limits_{e \in E} w(e)$ combined with the observation that any extreme point in this case is integral and therefore a perfect matching gives us the desired result.
`Alternate solution:` Note that we have an easier soultion for this if we assume the result of problem 4 of homework assignment 2 which tells us that the edges of this graph can be partitioned as k edge-disjoint perfect matchings. Suppose that each of the k matchings has weight more than $\frac{1}{k} \sum\limits_{e \in E} w(e)$, then the sum of weights of the matchings will be greater than $\sum\limits_{e \in E} w(e)$. This is a contradiction to the fact that the weight of the matchings should sum to be equal to the sum of edge weights. So we must have at least one perfect matching with weight less than or equal to $\frac{1}{k} \sum\limits_{e \in E} w(e)$. ■

Exercise 2: Recall that the matching of a bipartite graph $G = (V_L \cup V_R, E)$ is a subset of edges $F \subseteq E$ such that every vertex is adjacent to at most *one* edge in F. Give an algorithm that finds a subset of edges $F \subseteq E$ of maximum cardinality such that every vertex is adjacent to at most *two* edges in F.
`Note:` There is a slight modification in the question from that stated in the exercise sheet distributed in class, which said "subset of edges $F \subseteq E$ such that...". $F = \phi$ would work if F does not have to be of maximum cardinality.

Proof In the previous lecture, we formulated the bipartite matching problem as a matroid intersection. We will formulate this problem as a matroid intersection in an analogous, almost identical manner. Only difference being that degree constraints are now ≤ 2 instead of ≤ 1 as in the case of matchings. We define $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, both having the base set as the set of edges of G with:

$$\mathcal{I}_1 = \{S \subseteq E | \forall v \in V_L \quad |S \cap \delta(v)| \leq 2\}$$

$$\mathcal{I}_2 = \{S \subseteq E | \forall v \in V_R \quad |S \cap \delta(v)| \leq 2\}$$

where $\delta(v)$ is the set of edges incident to v in G. Here $\mathcal{I}_1$ corresponds to all edge collections of G such each vertex in V_L has at most 2 incident edges from each such collection. Similarly for $\mathcal{I}_2$ and V_R. We verify that M_1 is in fact a matroid. The proof for M_2 is analogous.
Let $S \in \mathcal{I}_1$ and $S' \subseteq S$. It is easy to see that $\forall v \in V_L$, $|S \cap \delta(v)| \leq 2 \implies |S' \cap \delta(v)| \leq 2$, i.e. $S \in \mathcal{I}_1 \implies S \in \mathcal{I}_1$. This verifies the first axiom. Now suppose $S_1, S_2 \in \mathcal{I}_1$ with $|S_1| > |S_2|$. This means that $\exists v \in V_L$ such that $|S_1 \cap \delta(v)| > |S_2 \cap \delta(v)|$. We consider $e \in (S_1 \cap \delta(v)) \backslash (S_2 \cap \delta(v))$, then $S_2 + e \in \mathcal{I}_1$. This proves the second axiom.
The maximum size independent set in $M_1 \cap M_2$ is exactly the set that we desire. We know from the result proved in the previous lecture that we can find a maximum cardinality independent set in a matroid intersection in polynomial time. We find F by this algorithm. ■

Exercise 3: Prove that $M \backslash e$ and M/e defined in definition 4 and 5 respectively are matroids.
Proof $M \backslash e$: Directly follows from the fact that M is a matroid since the independent sets of $M \backslash e$ are nothing but the independent sets of M which are subsets of $E - e$.
M/e: If $e \notin \mathcal{I}$, we are done as $\mathcal{I}' = \mathcal{I}$ in this case.
If $e \in \mathcal{I}$: Let $S' \subseteq S$. $S \in \mathcal{I}' \implies S + e \in \mathcal{I}$ by definition of M/e. Therefore, since $S' + e \subseteq S + e$, we get $S' + e \in \mathcal{I}$ which in turn gives us $S' \in \mathcal{I}'$. This proves the first axiom.
For the second axiom, let $S_1, S_2 \in \mathcal{I}'$ such that $|S_1| < |S_2|$.This implies $S_1 + e, S_2 + e \in \mathcal{I}$. Observe that $|S_1 + e| < |S_2 + e|$. Therefore $\exists f \in (S_2 + e) \backslash (S_1 + e) = S_2 \backslash S_1$ such that $A + e + f \in \mathcal{I} \implies A + f \in \mathcal{I}'$. This completes the proof. ■

2 Matroid Intersection Polytope

Definition 6 (Matroid Intersection Polytope) *For two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, define the matroid intersection polytope*

$$P(M_1 \cap M_2) = conv\{\mathcal{X}_I | I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$$

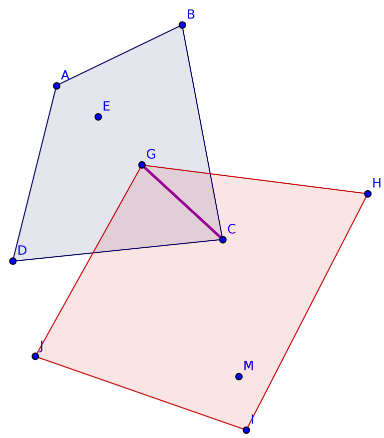
What follows is the main result of this lecture.

Theorem 7 (Matroid Intersection Polytope Theorem) (Edmonds)

$$P(M_1 \cap M_2) = P(M_1) \cap P(M_2)$$

This theorem says that the matroid intersection polytope of two matroids is the intersection of matroid polytopes of both the individual matroids.

`Remark:` This theorem seems intutive, but it is in fact a non-trivial result and not true in general. We illustrate this by an example. We consider two polytopes, the red and the blue one, each of them generated by a convex combination of a set of points. The polytope generated by their intersection is just the line GC in this case, which is marked purple and not the whole region included in the intersection of the polytopes.



So Theorem 7 states something quite strong and only holds because M_1 and M_2 are matroids. The rest of the lecture will be devoted to proving Theorem 7. As always, one direction is easy.

$P(M_1 \cap M_2) \subseteq P(M_1) \cap P(M_2)$: This inclusion follows from the definition. For every $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, $\mathcal{X}_I$ is in both $P(M_1)$ and $P(M_2)$.

`Observation:` Recall the definition of the matroid intersection polytope we gave in the previous lecture.

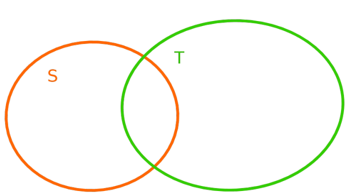
$$P(M_1) \cap P(M_2) = \{x \in \mathbb{R}^{|E|} \quad | \quad x(S) \leq r_1(S), x(S) \leq r_2(S) \quad \forall S \subseteq E \quad , \quad x_e \geq 0 \quad \forall e \in E\}$$

We observe that any $\{0,1\}$ solution to this system is the characteristic vector of a set in $\mathcal{I}_1 \cap \mathcal{I}_2$.

We need three lemmas before we can prove the other direction.

Definition 8 (Tight set) *Given a matroid M with rank function r and $x \in P(M)$ we say a set $S \subseteq E$ is a tight set of x with respect to r is $x(S) = r(S)$.*

Lemma 9 (Uncrossing operation) *Let $M = (E, \mathcal{I})$ be a matroid with rank function r. Let $x \in P(M)$. If S and T are tight sets of x with respect to r, so are $S \cup T$ and $S \cap T$.*



Proof Since $x \in P(M)$, we have $r(S \cup T) \geq x(S \cup T)$ and $r(S \cap T) \geq x(S \cap T)$. Therefore,

$$\begin{aligned} r(S \cup T) + r(S \cap T) &\geq x(S \cup T) + x(S \cap T) \\ &\overset{(a)}{=} x(S) + x(T) \\ &\overset{(b)}{=} r(S) + r(T) \\ &\overset{(c)}{\geq} r(S \cup T) + r(S \cap T) \end{aligned}$$

where (a) is due to linearity of $x(A) = \sum_{e \in A} x_e$, (b) is due to the tightness of S and T, and (c) is because r is submodular. Therefore, all inequalities must be equalities, which gives us $r(S \cup T) = x(S \cup T)$ and $r(S \cap T) = x(S \cap T)$. This proves the result. ■

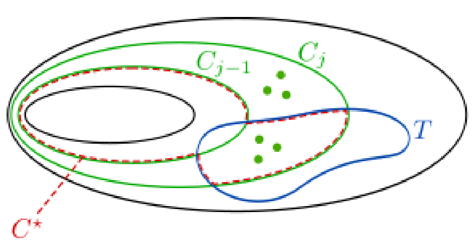
Lemma 10 *Let $M = (E, \mathcal{I})$ be a matroid with rank function r. Let $x \in P(M)$. Let $\mathcal{C} = \{C_1, C_2 \ldots C_k\}$ with $\phi \subset C_1 \subset C_2 \ldots \subset C_k$ be an inclusion-wise maximal chain of tight sets of x with respect to r. Then every tight set T os x with respect to r must satisfy $\mathcal{X}_T \in span\{\mathcal{X}_C : C \in \mathcal{C}\}$.*

Proof Suppose $\exists T$ such that $\mathcal{X}_T \notin span\{\mathcal{X}_C : C \in \mathcal{C}\}$
Case 1: $T \nsubseteq C_k$
Since both T and C_k are tight, $T \cup C_k$ is tight by *Lemma 9*. But this is a contradiction to $\mathcal{C}$ being maximal.
Case 2: $T \subseteq C_k$
We consider 'Increment sets' $C_j \backslash C_{j-1}$, with $C_0 = \phi$ by convention.
Claim: At least one of the incremental sets is cut by T, i.e. $(C_j \backslash C_{j-1}) \cap T \neq \phi$ and $(C_j \backslash C_{j-1}) \backslash T \neq \phi$.
Proof Suppose not. Then we can write $T = \bigcup_{j \in \mathcal{J}} (C_j \backslash C_{j-1})$ for some $\mathcal{J} \subseteq \{1, 2 \ldots k\}$. This implies

$$\mathcal{X}_T = \sum_{j \in \mathcal{J}} \mathcal{X}_{(C_j \backslash C_{j-1})} = \sum_{j \in \mathcal{J}} (\mathcal{X}_{C_j} - \mathcal{X}_{C_{j-1}})$$

But this contradicts the assumption that $\mathcal{X}_T \notin span\{\mathcal{X}_C : C \in \mathcal{C}\}$. This proves our claim. ■

Therefore $T \cap (C_j \backslash C_{j-1})$ is a proper subset of $C_j \backslash C_{j-1}$ for some j and we must have the situation illustrated in the figure below:



$C^* = (T \cap C_j) \cup C_{j-1}$ is tight by *Lemma 9*. By construction $C_{j-1} \subset C^* \subset C_j$, which is a contradiction to maximality of $\mathcal{C}$.
We have thus shown that for any tight set T, if $\mathcal{X}_T \notin span\{\mathcal{X}_C : C \in \mathcal{C}\}$ we can have neither $T \nsubseteq C_k$ nor $T \subseteq C_k$. This proves that $\mathcal{X}_T \in span\{\mathcal{X}_C : C \in \mathcal{C}\}$ which is the desired result. ■

Definition: $\text{supp}(x) = \{e \in E | x_e > 0\}$

Lemma 11 *Let M_1 and M_2 be two matroids over the same ground set with rank functions r_1 and r_2. If x is an extreme point solution of $P(M_1) \cap P(M_2)$, then there exist two chains*

$$\begin{aligned} \mathcal{C} &= \{C_1, C_2 \dots C_k\} \quad \textit{with} \quad \phi \subset C_1 \subset C_2 \dots C_k \\ \mathcal{D} &= \{D_1, D_2 \dots D_l\} \quad \textit{with} \quad \phi \subset D_1 \subset D_2 \dots D_l \end{aligned}$$

such that

1. $x(C_i) = r_1(C_i) \quad \forall \quad C_i$
   $x(D_j) = r_2(D_j) \quad \forall \quad D_j$
2. $\{\mathcal{X}_C | C \in \mathcal{C} \cup \mathcal{D}\}$ *contains at least —supp(x)— linearly independent vectors.*

Proof We construct $\mathcal{C}$ and $\mathcal{D}$ as inclusion-wise maximal chains of tight sets of x with respect to r_1 and r_2 respectively. The first part of the lemma is therefore true by construction. Recall that

$$P(M_1) \cap P(M_2) = \left\{ x \geq 0 : \begin{array}{ll} x(S) \leq r_1(S) & \forall S \subseteq E \\ x(S) \leq r_2(S) & \forall S \subseteq E \end{array} \right\}$$

Since x is an extreme point of this polytope, it must have $|E|$ linearly independent tight constraints. Exactly $|E| - |supp(x)|$ of them are the constraints $x_e = 0$. So the tight constraints of the type $x(S) = r_1(S)$ and $x(S) = r_2(S)$ have rank at least $|supp(x)|$. On the other hand by *lemma 10* we have that $span\{x_C | C \in \mathcal{C}\}$ contains the characteristic vector $\mathcal{X}_T$ of all tight sets T of x with respect to r_1. Likewise for $\mathcal{D}$ and r_2. It follows that $span\{x_C | C \in \mathcal{C} \cup \mathcal{D}\}$ contains the characteristic vectors $\mathcal{X}_T$ of all tight sets T of x with respect to r_1 and r_2. Therefore the two chains span a subspace of dimension at least $|supp(x)|$, which proves the second part of the lemma. ■

We now have all the tools necessary to prove the main theorem

Proof of theorem 7

Proof We prove $P(M_1 \cap M_2) \supseteq P(M_1) \cap P(M_2)$ by induction on E. It is trivial for $|E| = 0$. Assume that x is an extreme point of $P(M_1) \cap P(M_2)$ that is not in $P(M_1 \cap M_2)$.

Case 1: $x_e = 0$ for some e

Delete e from M_1 and M_2 to obtain $\widetilde{M_1}$ and $\widetilde{M_2}$. Restrict x to $E - e$ to obtain $\tilde{x} \in \mathbb{R}^{E-e}$. So we have $\tilde{x} \in P(\widetilde{M_1})$ and $\tilde{x} \in P(\widetilde{M_2})$. By induction hypothesis $\tilde{x} \in P(\widetilde{M_1}) \cap P(\widetilde{M_2})$, which implies $x \in P(M_1 \cap M_2)$.

Case 2: $x_e = 1$ for some e

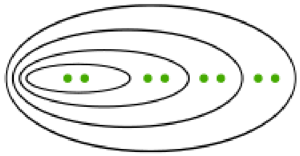
Contract e from M_1 and M_2 to obtain $\widetilde{M_1}$ and $\widetilde{M_2}$. Restrict x to $E - e$ to obtain $\tilde{x} \in \mathbb{R}^{E-e}$. We have $\tilde{x} \in P(\widetilde{M_1})$ since

$$\forall S \subseteq E - e, \quad r_{\tilde{M}_1}(S) = r_1(S + e) - 1 \geq x(S + e) - 1 = \tilde{x}(S)$$

Likewise, $\tilde{x} \in P(\widetilde{M_2})$. By induction hypothesis $\tilde{x} \in P(\widetilde{M_1}) \cap P(\widetilde{M_2})$. Therefore $\tilde{x}$ is a convex combination of characteristic vectors of independent sets in $\widetilde{\mathcal{I}_1} \cap \widetilde{\mathcal{I}_2}$. Adding e to independent sets in $\widetilde{M_1}$ and $\widetilde{M_1}$ gives us independent sets in M_1 and M_2. Therefore x is a convex combination of characteristic vectors of independent sets in $\mathcal{I}_1 \cap \mathcal{I}_2$, which is the same as saying $x \in P(M_1 \cap M_2)$.

Case 3: $\forall e \quad 0 < x_e < 1$

This is the most interesting case, because it does not occur! Consider $\mathcal{C}$ and $\mathcal{D}$ defined by *Lemma 11*. Since $x(C_i) = x(r_i) \in \mathbb{Z}$ and every x_e is fractional, each increment set must contain at least two elements to ensure integer sums $x(C_i)$, as illustrated in the figure below



Therefore $\mathcal{C} \leq |E|/2$ and $\mathcal{D} \leq |E|/2$. *Lemma 11* states that $\{\mathcal{X}_C | C \in \mathcal{C} \cup \mathcal{D}\}$ contains $\text{supp}(x) = |E|$ linearly independent vectors. Therefore, we must have $|\mathcal{C}| = |\mathcal{D}| = |E|/2$. This implies that that the maximal sets in the two chains must be $C_k = D_l = E$, i.e., $\mathcal{C} \cup \mathcal{D}$ contains at most $|E| - 1$ sets, a contradiction. This completes the proof. ■

`Remark:` We note that *Theorem 7* implies an algorithm for matroid intersection in the form os solving a linear program. Namely, to maximize $\sum_e w_e x_e$ under the constraints

$$\begin{aligned} x(S) &\leq r_1(S) \quad \forall S \subseteq E \\ x(S) &\leq r_2(S) \quad \forall S \subseteq E \\ x_e &\geq 0 \qquad\quad \forall e \in E \end{aligned}$$

3 Matroid Union

Definition 12 *Given matroids* $M_1, \ldots M_k = (E_k, \mathcal{I}_k)$

$$M_1 \vee \ldots \vee M_k = (E_1 \cup \ldots E_k, \mathcal{I}) \quad with \quad \mathcal{I} = \{I_1 \cup \ldots \cup I_k | I_i \in \mathcal{I}_i\}$$

`Remark:` Note that $E_1, \ldots E_k$ need not be disjoint and can be interecting. Matroid union and matroid intersection are closely related in this case. Also, $M_1 \vee \ldots \vee M_k$ is a matroid.

Exercise 4: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs, i.e., $V_1 \cap V_2 = \phi$. Consider their graphic matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$. It is easy to see that $M = (E_1 \cup E_2, \mathcal{I})$ with

$$\mathcal{I} = \{I_1 \cup I_2 : I_i \in \mathcal{I}_i\}$$

is a matroid: for any $F_1 \cup F_2 \subseteq E_1 \cup E_2$ with $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$, $(V_1 \cup V_2, F_1 \cup F_2)$ is acyclic if and only if both (V_1, E_1) and (V_2, E_2) are acyclic.
Prove the following generalisation of this observation: let $M_1, \ldots M_k = (E_k, \mathcal{I}_k)$ be matroids defined on mutually disjoint ground sets. Prove that $M = (E_1 \cup \ldots E_k, \mathcal{I})$ with

$$M_1 \vee \ldots \vee M_k = (E_1 \cup \ldots E_k, \mathcal{I}) \quad with \quad \mathcal{I} = \{I_1 \cup \ldots \cup I_k | I_i \in \mathcal{I}_i\}$$

is also a matroid.
Proof Closedness is easy to see.
Let $A, B \in \mathcal{I}$ with $|A| < |B|$. This implies that $\exists i$ such that $|A \cap E_i| < |B \cap E_i|$. Therefore $\exists f \in (B \cap E_i) \backslash (A \cap E_i)$ such that $(A \cap E_i) + f \in \mathcal{I}_i$ and therefore $A + f \in \mathcal{I}$. ■

Theorem 13 $M_1 \vee \ldots \vee M_k$ *is a matroid.*

Proof We treat the duplicate elements as different and apply the result of Exercise 4. This gives a matroid. ■

Lemma 14 *Given a matroid $M' = (E', \mathcal{I}')$ with rank function r', and a function $f : E' \to E$,*

$$\mathcal{I} = \{f(I') : I' \in \mathcal{I}'\}$$

defines a matroid.

Proof It is easy to see that $\mathcal{I}$ is closed under subsets. We will prove the extension axiom by induction. Let $A, B \in \mathcal{I}, |A| < |B|$. Choose $A', B' \in \mathcal{I}'$ such that $f(A') = A, f(B') = B, |A'| = |A|, |B'| = |B|$ and $|A' \cap B'|$ as large as possible. Since M' is a matroid, $\exists e' \in B' \setminus A'$ such that $A' + e' \in \mathcal{I}'$. If $f(e') = e \in A = f(A')$ then e has another pre-image a in A'. Note that $a \notin B'$ because f is a bijection on B'. However, we could take $A" = A - a + e'$ instead of A', thus increasing $|A' \cap B'|$, which is a contradiction to having chosen a maximal such set. Therefore, $e \in B \setminus A$, which proves the result. ∎

[2] gives a proof of *Theorem 13* using *lemma 14*.

References

[1] Jan Vondrák *Lecture 10 Notes of the course Polyhedral techniques in combinatorial optimization* October 21, 2010.

[2] Jan Vondrák *Lecture 11 Notes of the course Polyhedral techniques in combinatorial optimization* October 28, 2010.