ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Homework I, Topics in Theoretical Computer Science 2014

Due on Tuesday March 11 at 14:00 (send an email to ola.svensson@epfl.ch). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups with up to three students, but solutions should be handed in individually and please write with whom you have collaborated.

1 (20p, problem 5.8.3 from the book "the probabilistic method")

Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$.

2 (30p, adapted from problem 5.8.4 from the book "the probabilistic method") **Updated (simplified).**

Let $G = (V, E)$ be a cycle of length $11n$ and let $V = V_1 \cup V_2 \cup \ldots \cup V_n$ be a partition of its $11n$ vertices into n pairwise disjoint subsets, each of cardinality 11. Prove that there must be an independent set of G containing precisely one vertex from each V_i.

3 (50 p, from Johan Håstad's course "Theoreticians toolkit" at KTH)

Constructing a random 3-SAT formula with n variables and $m = \lceil dn \rceil$ (remember that $\lceil x \rceil$ is the smallest integer larger than x) clauses is done in the following way. Randomly take three different variables (all triples being equally likely). With uniform probability choose one of the eight ways to negate these variables and make them into a clause. Repeat with independent randomness until you have m clauses.

3a (20p) For what value of d is the expected number of satisfying assignments roughly[1] 1? Call this value d_0.

3b (5p) Prove that the formula is likely (probability $1 - o(1)$) to be unsatisfiable for any constant d such that $d > d_0$.

3c (25p) Prove that the formula remains at least somewhat likely to be unsatisfiable also in the case when d is slightly smaller than d_0. The difficulty of this problem is very much dependent on what we mean by "somewhat likely" and "slightly smaller". The exact formulation to prove to get a full score on this problem is that there is some constant $d_1 < d_0$ such that for $d = d_1$ the probability that the corresponding random formula is satisfiable is at most $1/2$. The size of $d_0 - d_1$ does not matter for your score on the problem and the main property of a solution to aim for is a mathematically correct argument.

Hint: A satisfiable formula that does not depend on all its variables has many satisfying assignments.

As a curiosity we may note that the value of d such that such a formula has probability around $1/2$ of being satisfiable is not known but conjectured to be around 4.2.

[1] To be more precise $\Theta(1)$.