ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Homework II, Topics in Theoretical Computer Science 2015

Due on Monday March 30 at 09:15 (send an email to ola.svensson@epfl.ch). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups with up to three students. On this problem set the first problem is about finding information and here you should feel free to use any source.

1 *(25 pts)* Prove the Strong Duality Theorem for Linear Programming. You are allowed to use any source for this exercise but the proof should be written using your own words. Also please cite the reference that you used. (Many proofs use Farkas' Lemma which you should explain if you use it but not necessarily prove.)

2 *(25 pts)* Consider the Maximum Disjoint Paths problem: given an undirected graph $G = (V, E)$ with designated source $s \in V$ and sink $t \in V \setminus \{s\}$ vertices, find the maximum number of edge-disjoint paths from s to t. To formulate it as a linear program, we have a variable x_p for each possible path p that starts at the source s and ends at the sink t. The intuitive meaning of x_p is that it should take value 1 if the path p is used and 0 otherwise[1]. Let P be the set of all such paths from s to t. The linear programming relaxation of this problem now becomes

$$\begin{aligned} \text{Maximize} \quad & \sum_{p \in P} x_p \\ \text{subject to} \quad & \sum_{p \in P : e \in p} x_p \leq 1, \qquad \forall e \in E, \\ & x_p \geq 0, \qquad \forall p \in P. \end{aligned}$$

What is the dual of this linear program? What famous combinatorial problem do binary solutions to the dual solve?

[1] I know that the number of variables may be exponential, but let us not worry about that.

3 *(25 pts)* Consider the Maximum Weight Spanning Tree Linear Program:

$$\begin{aligned}
\text{Maximize} \sum_{e\in E} w(e)x_e & \\
\text{subject to} \quad \sum_{e\in E} x_e = |V|-1, & \\
\sum_{e\in E: e\subseteq S} x_e \le |S|-1 & \qquad \forall S\subset V: |S|\ge 2,\\
x_e \ge 0 & \qquad \forall e\in E.
\end{aligned}$$

The dual linear program can be written as follows:

$$\begin{aligned}
\text{Minimize} \sum_{S\subseteq V: |S|\ge 2} y_S(|S|-1) & \\
\text{subject to} \sum_{S: e\subseteq S} y_S \ge w(e), & \\
y_S \ge 0 & \qquad \forall S\subset V: 2\le |S| \le |V|-1.
\end{aligned}$$

Note that in the dual the variable y_V can take both positive and negative values whereas the other variables are restricted to be non-negative.

Adapt the primal-dual algorithm seen in class so that it also works in the presence of negative edge-weights. Note that this shows that the above linear program for maximum weight spanning tree is integral.

4 *(25 pts)* Use the integrality of the biparite perfect matching polytope (as proved in class) to show the following classical result:

> "The edge set of a k-regular bipartite graph $G = (A\cup B, E)$ can in polynomial time be partitioned into k-disjoint perfect matchings"

A graph is k-regular if the degree of each vertex equals k. Two matchings are disjoint if they do not share any edges.