1 (25 pts) In these two subproblems, we shall study circuits of bounded depth. The depth of a circuit is the length of the longest directed path from an input to the output gate.

1a (10 pts) Show that any language \( L \) can be decided by a circuit family \( \{C_n\}_{n \in \mathbb{N}} \) where \( C_n \) has constant depth but is allowed to have unbounded fan-in and exponential size. More specifically, \( C_n \) can be constructed using three layers: the first layer consists of (potential) NOT gates of the input bits, the second of AND gates (of unbounded fan-in), and the last layer is a single OR gate (of unbounded fan-in).

1b (15 pts) Let \( \text{PARITY} = \{ x \in \{0,1\}^* : \oplus_{i \in |x|} x_i = 1 \pmod{2} \} \) be the language consisting of strings of odd parity. Show that \( \text{PARITY} \) cannot be decided by a circuit family \( \{C_n\}_{n \in \mathbb{N}} \) where \( C_n \) has depth \( o(\log n) \) and the maximum fan-in of any gate is two.

(As a side remark, we note the following much stronger result by Håstad: \( \text{PARITY} \) cannot be decided by circuits that have depth \( o(\log n / \log \log n) \), unbounded fan-in, and are of polynomial size.)

2 (27 pts, Exercise 5.10 from the textbook) Suppose \( A \) is some language such that \( \text{P}^A = \text{NP}^A \). Then show that \( \text{PH}^A \subseteq \text{P}^A \).

Hint: First study the proof of “\( \text{P} = \text{NP} \) implies \( \text{PH} = \text{P} \)” (Theorem 5.4 in the textbook). Then show that this proof relativizes, i.e., holds with respect to any oracle.
3 (48 pts, Exercise from Luca Trevisan’s course) Suppose that there is a deterministic polynomial-time algorithm $A$ that on input (the description of) a circuit $C$ produces a number $A(C)$ such that

$$\Pr_x[C(x) = 1] - \frac{2}{5} \leq A(C) \leq \Pr_x[C(x) = 1] + \frac{2}{5}.$$ 

3a (22 pts) Prove that it follows that $P = BPP$. 

3b (26 pts) Prove that there exists a deterministic algorithm $A'$ that, on input a circuit $C$ and a parameter $\epsilon$, runs in time polynomial in the size of $C$ and in $1/\epsilon$ and produces a value $A'(C, \epsilon)$ such that

$$\Pr_x[C(x) = 1] - \epsilon \leq A'(C, \epsilon) \leq \Pr_x[C(x) = 1] + \epsilon.$$ 

Hint: The threshold function $f_t : \{0, 1\}^n \rightarrow \{0, 1\}$, where $t \in \{1, \ldots, n\}$, defined by

$$f(x) = 1 \Leftrightarrow \sum_i x_i \geq t$$

has a polynomial-sized (in $n$) circuit.