ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Homework III, Topics in Theoretical Computer Science 2015

Due on Tuesday April 21 at 12:00 AM (send an email to ola.svensson@epfl.ch). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups with up to three students.

1 *(25 pts)* Consider the following problem where we wish to select a subset of the edges/arcs so that the given targets of the in-degrees and the out-degrees are satisfied:

Input: A digraph $G = (V, E)$ and, for each vertex $v \in V$, a bound $i(v) \in \mathbb{N}$ on its in-degree and a bound $o(v) \in \mathbb{N}$ on its out-degree.

Output: If possible, a subset $E' \subseteq E$ of the arcs satisfying

$$|\delta^-_{E'}(v)| = i(v) \qquad \text{and} \qquad |\delta^+_{E'}(v)| = o(v) \qquad \text{for each } v \in V,$$

where $\delta^-_{E'}(v)$ and $\delta^+_{E'}(v)$ denote the set of edges in E' that are directed towards v and the set of those edges in E' that are directed out of v, respectively.

Otherwise, if no such set E' exists, output a short certificate/proof that there is no solution to the problem.

Design and analyze an efficient algorithm for the above problem. You are allowed to use algorithms and results seen in class without reexplaining them.

2 *(25 pts)* Consider a bipartite graph $G = (A \cup B, E)$. We say that a subset $S \subseteq A$ of the vertices on the left-hand-side is *good* if there exists a matching M so that all nodes in S are matched. Prove that $M = (A, \mathcal{I})$ is a matroid where

$$\mathcal{I} = \{S \subseteq A : S \text{ is good}\}.$$

Note that in the above matroid the ground set is A and *not* the set of edges.

(*Hint: To prove this, it is good to remember the augmenting path algorithm for bipartite matching that we saw in class*)

3 *(50 pts)* Consider a k-regular k-edge connected undirected graph $G = (V, E)$. That is, each vertex has degree k and at least k edges need to be removed in order to disconnect the graph.

3a *(25 pts)* For any nonnegative edge weights $w : E \to \mathbb{R}_+$, show that there is a spanning tree T such that

$$\sum_{e \in T} w(e) < \frac{2}{k} \sum_{e \in E} w(e).$$

3b *(25 pts)* Show that G has a spanning tree T in which at least $\frac{|V|}{k+1}$ vertices have degree exactly 2.

(Hint: In both problems, the spanning tree polytope will be useful. For the second problem, the matroid intersection polytope is a good tool (the graphic matroid + another matroid).)