ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Homework IV, Topics in Theoretical Computer Science 2014

Due on Friday May 16 at 17:00 (send an email to ola.svensson@epfl.ch or leave it in my letter box outside my office INJ 112). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups with up to three students; it is sufficient to hand in one solution per group.

1 (25 p) Consider the standard linear programming relaxation of Set Cover that we saw in class.

1a (10 p) We gave a randomized rounding algorithm for the Set Cover problem. Use similar techniques to give an algorithm that with constant probability returns a collection of sets that cover at least half the elements and has cost at most a constant factor larger than the LP solution.

1b (15 p) Let f be a fixed constant, and consider instances of Set Cover where each element can be covered by at most f sets. We proved in class that the integrality gap of the LP is upper bounded by f for these instances. Provide examples to show that this bound is essentially tight.

(This is exercise 15.3 in the book of Vazirani[1] where you can also find a hint if needed.)

2 (30 p) In a Vertex Cover instance $G = (V, E)$ an edge $\{u, v\} \in E$ equals the constraint that either u or v must be picked. For some applications it is undesirable to pick both so we need to also introduce a second type, called exclusive-or edges, that requires us to pick exactly one of the two incident vertices.

Give a polynomial time algorithm for the generalization of Vertex Cover, where we have both ordinary and exclusive-or edges, that verifies if a solution exists and if it exists returns a 2-approximate solution.

(*Hint: This problem can be solved using LP rounding but it is much easier to solve it using SDP rounding.*)

[1] The book is available here: *http://www.cc.gatech.edu/fac/Vijay.Vazirani/book.pdf*

3 (45 p) Let $G = (V, E)$ be a bipartite graph. In class we saw that the following natural linear programming relaxation of the maximum weight matching problem in G is integral, i.e., every extreme point solution is integral.

$$\begin{array}{llcll} \text{maximize} & \sum_{e\in E} x_e & & & \\ \text{subject to} & \sum_{e\in E: v\in e} x_e & \leq & 1 & \forall v\in V \\ & x_e & \geq & 0 & \forall e \in E \end{array}$$

3a (20 p) Use this fact to prove the following classical result:

"The edge set of a k-regular bipartite graph $G = (V_1 \cup V_2, E)$ can in polynomial time be partitioned into k-disjoint matchings".

A graph is k-regular if the degree of each vertex equals k. Two matchings are disjoint if they do not share any edges.

3b (20 p) Obtain the dual of the above LP and show that it is an exact LP-relaxation (i.e., always has an integral optimal solution) for the problem of finding a minimum vertex cover in bipartite graph G.

3c (5 p) Use the previous result to derive the König-Egerváry Theorem:

$$\text{“}\textit{In any bipartite graph,} \max_{\textit{matching } M} |M| = \min_{\textit{vertex cover } U} |U|.\text{”}$$