ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Bonus Homework, Approximation Algorithms and Hardness of Approximation 2013

Due on Thursday June 6 at 23:59 (send an email to ola.svensson@epfl.ch). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups with up to three students, but solutions should be handed in individually and please write with whom you have collaborated.

1 (60 pts) In this exercise we shall complete the proof that MAX3-SAT is NP-hard to approximate within $7/8 + \epsilon$ for every $\epsilon > 0$. We start by finalizing the calculations of Håstad's verifier and then we shall prove that the verifier indeed implies the stated hardness for MAX3-SAT. First, recall the steps taken by Håstad's linear verifier V_H that performs 3 binary queries:

HÅSTAD'S VERIFIER V_H
Input: A 2CSPW instance where we expect a proof $\tilde{\pi}$ to encode the assignment to each variable $i \in [n]$ by using the folded long-code, i.e., if i is assigned label $w \in [W]$ then the proof "should" contain for variable i the truth-table of the function $\chi_w : \{\pm 1\}^W \to \{\pm 1\}$ defined by $\chi_w(x) = x_w$.
Goal: Do 3 binary queries and do a linear test that checks with "good" probability if the proof $\tilde{\pi}$ corresponds to a satisfying assignment or if it is far from a satisfying assignment.

- Pick a random constraint of the 2CSPW instance with projection function $h : [W] \to [W]$ and let $f, g : \{\pm 1\}^W \to \{\pm 1\}$ be the functions encoding the assignment of the two variables of the constraint (the projection h is from the variable represented by f to the variable represented by g).
- Pick $x, y \in \{\pm 1\}^W$ independently and uniformly at random.
- Pick $z \in \{\pm 1\}^W$ by setting each coordinate i independently as follows:

$$z_i = \begin{cases} 1 & \text{with probability } 1 - \gamma \\ -1 & \text{with probability } \gamma \end{cases}.$$

- Accept iff
$$f(x)g(y) = f(x\mathcal{H}^{-1}(y)z),$$
where $\mathcal{H}^{-1}(y)$ is the vector such that $\mathcal{H}^{-1}(y)_w = y_{h(w)}$.

1a (20 pts) Show that the acceptance probability of V_H equals

$$\mathbb{E}_{(f,g,h)}\left[\frac{1 + \sum_{S \subseteq [W]} \hat{f}_S^2 \hat{g}_{h_2(S)} (1-2\gamma)^{|S|}}{2}\right],$$

where the expectation is over the randomly picked constraint (f, g, h) and the function $h_2 : [W] \to [W]$ is defined as in class, i.e.,

$$h_2(S) = \{u \in W : |h^{-1}(u) \cap S| \text{ is odd}\}.$$

1b (20 pts) Use the above characterization of the acceptance probability to show the following: If V_H accepts with probability $1/2 + \delta$ (for some $\delta \geq 0$) then

$$\mathbb{E}_{(f,g,h)}\left[\sum_{S \subseteq [W]} \frac{\hat{f}_S^2 \hat{g}_{h_2(S)}^2}{|S|}\right] \geq \gamma\delta^2 \tag{1}$$

Hint: Use Cauchy-Schwarz' inequality, Parseval's identity, and $\frac{1}{x} \geq e^{-x}$ for $x > 0$.

1c (20 pts) In class we gave a randomized "decoding" that showed that the 2CSPW instance always have an assignment that satisfies at least (1) fraction of the constraints and therefore the soundness of V_H can be made arbitrarily close to 1/2. We have thus completed the proof of the theorem:

> HÅSTAD'S 3-BIT **PCP**
> For every $\delta > 0$ and every language $L \in \mathbf{NP}$, there is **PCP**-verifier V_H for L that reads $O(\log n)$ random bits and makes three binary queries so that given an input x
>
> - If $x \in L$ there exists a proof that makes V_H accept with probability $1 - \delta$.
> - IF $x \notin L$ then V_H accepts any proof with probability at most $1/2 + \delta$.
>
> Moreover, the test made by V_H is linear, i.e., given a proof π, V_H chooses a triple (i_1, i_2, i_3) and $b \in \{0, 1\}$ according to some distribution and accepts iff $\pi_{i_1} + \pi_{i_2} + \pi_{i_3} = b \mod 2$.

Your task is now to show that the above PCP implies that it is NP-hard to approximate MAX3-SAT within $7/8 + \epsilon$ for any fixed $\epsilon > 0$.

Hint: there is a simple gadget that transforms a linear equation of the form $\pi_{i_1} + \pi_{i_2} + \pi_{i_3} = b$ to four 3SAT clauses that are all satisfied iff the equation is satisfied.

2 (30 pts) You are given a positive integer q and a set of equations in the form $x_i - x_j = c_{ij} \pmod q$. Define G to be a graph with a vertex i for each variable x_i and an edge (i, j) if there is an equation involving variables x_i and x_j. Suppose G is a 3-regular graph. Our goal is to find an integral assignment for each variable $x_i \in [0, q)$ that maximizes the number of satisfied equations. Give a $\frac{2}{3}$-approximation algorithm for this problem.

3 (30 pts, "counter example to (simple) parallel repetition" see Problem 22.6(a) in the book of Arora and Barak)

Consider the following 2CSP instance φ on an alphabet of size 4 (which we identify with $\{0,1\}^2$). The instance φ has four variables $x_{0,0}, x_{0,1}, x_{1,0}, x_{1,1}$ and four constraints $C_{0,0}, C_{0,1}, C_{1,0}$, $C_{1,1}$. The constraint $C_{a,b}$ looks at the variables $x_{0,a}$ and $x_{1,b}$ and outputs TRUE if and only if $x_{0,a} = x_{1,b}$ and $x_{0,a} \in \{0a, 1b\}$.

Prove that $\text{val}(\varphi^{*2}) = \text{val}(\varphi)$, where φ^{*t} denotes the 2CSP over alphabet W^t that is the t-times parallel repeated version of φ (see Section 22.3.1 in the book of Arora and Barak that is available as a pdf following a link from the course page).