1 (60 pts) In this exercise we shall complete the proof that MAX3-SAT is NP-hard to approximate within $\frac{7}{8} + \epsilon$ for every $\epsilon > 0$. We start by finalizing the calculations of Håstad’s verifier and then we shall prove that the verifier indeed implies the stated hardness for MAX3-SAT. First, recall the steps taken by Håstad’s linear verifier $V_H$ that performs 3 binary queries:

**Håstad’s verifier $V_H$**

*Input:* A 2CSPW instance where we expect a proof $\tilde{\pi}$ to encode the assignment to each variable $i \in [n]$ by using the folded long-code, i.e., if $i$ is assigned label $w \in [W]$ then the proof “should” contain for variable $i$ the truth-table of the function $\chi_w : \{\pm 1\}^W \rightarrow \{\pm 1\}$ defined by $\chi_w(x) = x_w$.

*Goal:* Do 3 binary queries and do a linear test that checks with “good” probability if the proof $\tilde{\pi}$ corresponds to a satisfying assignment or if it is far from a satisfying assignment.

- Pick a random constraint of the 2CSPW instance with projection function $h : [W] \rightarrow [W]$ and let $f, g : \{\pm 1\}^W \rightarrow \{\pm 1\}$ be the functions encoding the assignment of the two variables of the constraint (the projection $h$ is from the variable represented by $f$ to the variable represented by $g$).
- Pick $x, y \in \{\pm 1\}^W$ independently and uniformly at random.
- Pick $z \in \{\pm 1\}^W$ by setting each coordinate $i$ independently as follows:
  $$z_i = \begin{cases} 
  1 & \text{with probability } 1 - \gamma \\
  -1 & \text{with probability } \gamma 
  \end{cases}$$
- Accept iff
  $$f(x)g(y) = f(xh^{-1}(y)z),$$
  where $h^{-1}(y)$ is the vector such that $h^{-1}(y)_w = y_{h(w)}$.

1a (20 pts) Show that the acceptance probability of $V_H$ equals

$$\mathbb{E}_{(f,g,h)} \left[ \frac{1 + \sum_{S \subseteq [W]} \hat{f}_S^2 \hat{g}_h(S)(1 - 2\gamma)^{|S|}}{2} \right],$$

where the expectation is over the randomly picked constraint $(f, g, h)$ and the function $h_2 : [W] \rightarrow [W]$ is defined as in class, i.e.,

$$h_2(S) = \{ u \in W : h^{-1}(u) \cap S \text{ is odd} \}.$$
1b (20 pts) Use the above characterization of the acceptance probability to show the following:
If $V_H$ accepts with probability $1/2 + \delta$ (for some $\delta \geq 0$) then
\[
E_{(f,g,h)} \left[ \sum_{S \subseteq [W]} \frac{\hat{f}_S^2 \hat{g}_S^2 \hat{h}_S(S)}{|S|} \right] \geq \gamma \delta^2
\]  
(1)

Hint: Use Cauchy-Schwarz’ inequality, Parseval’s identity, and $\frac{1}{x} \geq e^{-x}$ for $x > 0$.

1c (20 pts) In class we gave a randomized "decoding" that showed that the 2CSP-W instance always have an assignment that satisfies at least $\frac{1}{2}$ fraction of the constraints and therefore the soundness of $V_H$ can be made arbitrarily close to $1/2$. We have thus completed the proof of the theorem:

**Håstad’s 3-bit PCP**

For every $\delta > 0$ and every language $L \in \text{NP}$, there is $\text{PCP}$-verifier $V_H$ for $L$ that reads $O(\log n)$ random bits and makes three binary queries so that given an input $x$

- If $x \in L$ there exists a proof that makes $V_H$ accept with probability $1 - \delta$.
- If $x \not\in L$ then $V_H$ accepts any proof with probability at most $1/2 + \delta$.

Moreover, the test made by $V_H$ is linear, i.e., given a proof $\pi$, $V_H$ chooses a triple $(i_1, i_2, i_3)$ and $b \in \{0, 1\}$ according to some distribution and accepts iff $\pi_{i_1} + \pi_{i_2} + \pi_{i_3} = b \mod 2$.

Your task is now to show that the above PCP implies that it is NP-hard to approximate MAX3-SAT within $7/8 + \epsilon$ for any fixed $\epsilon > 0$.

Hint: there is a simple gadget that transforms a linear equation of the form $\pi_{i_1} + \pi_{i_2} + \pi_{i_3} = b$ to four 3SAT clauses that are all satisfied iff the equation is satisfied.

2 (30 pts) You are given a positive integer $q$ and a set of equations in the form $x_i - x_j = c_{ij} \pmod{q}$. Define $G$ to be a graph with a vertex $i$ for each variable $x_i$ and an edge $(i, j)$ if there is an equation involving variables $x_i$ and $x_j$. Suppose $G$ is a 3-regular graph. Our goal is to find an integral assignment for each variable $x_i \in \{0, q\}$ that maximizes the number of satisfied equations. Give a $\frac{2}{3}$-approximation algorithm for this problem.

3 (30 pts, “counter example to (simple) parallel repetition” see Problem 22.6(a) in the book of Arora and Barak)

Consider the following 2CSP instance $\varphi$ on an alphabet of size 4 (which we identify with $\{0, 1\}^2$). The instance $\varphi$ has four variables $x_{0,0}, x_{0,1}, x_{1,0}, x_{1,1}$ and four constraints $C_{0,0}, C_{0,1}, C_{1,0}, C_{1,1}$. The constraint $C_{a,b}$ looks at the variables $x_{0,a}$ and $x_{1,b}$ and outputs TRUE if and only if $x_{0,a} = x_{1,b}$ and $x_{0,a} \in \{0a, 1b\}$.

Prove that $\text{val}(\varphi^{\ast t}) = \text{val}(\varphi)$, where $\varphi^t$ denotes the 2CSP over alphabet $W^t$ that is the $t$-times parallel repeated version of $\varphi$ (see Section 22.3.1 in the book of Arora and Barak that is available as a pdf following a link from the course page).